Learning Deterministic One-Counter Automata

OL: Polynomial-time active-learning algorithm for DOCA*

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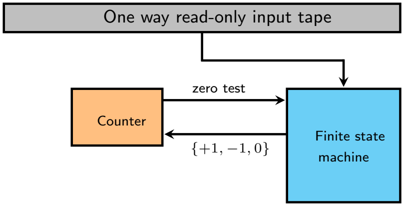
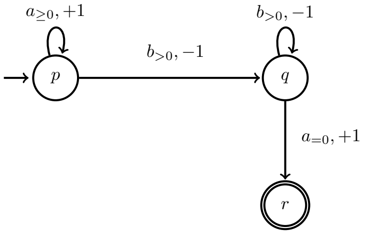
One counter automata



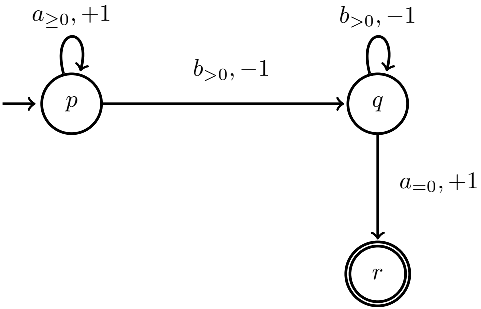
Counter: Can be incremented, decremented or tested for zero.

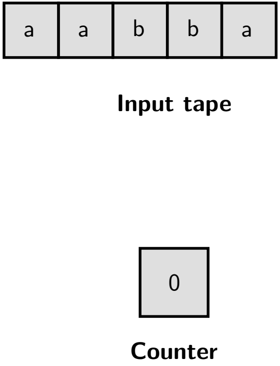
DOCA: Deterministic One Counter Automata.



OCA accepting $\{a^n b^n a \mid n \geq 0\}$.

Example: $a^n b^n a$





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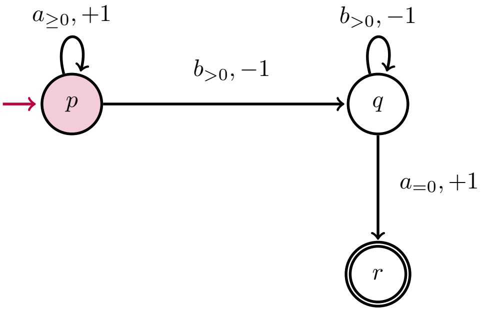

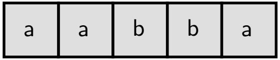

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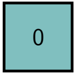

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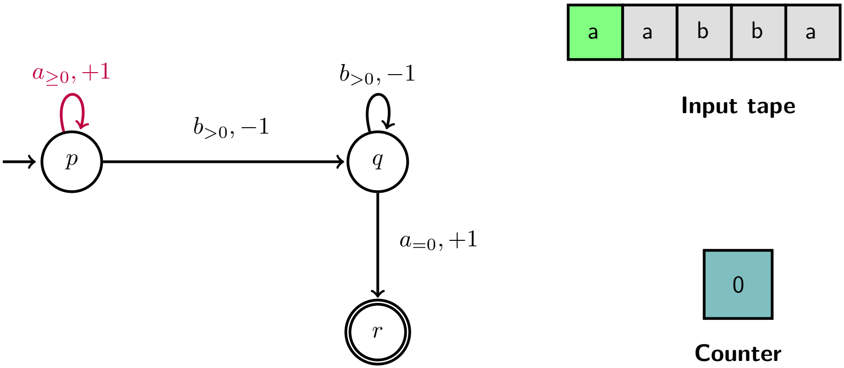

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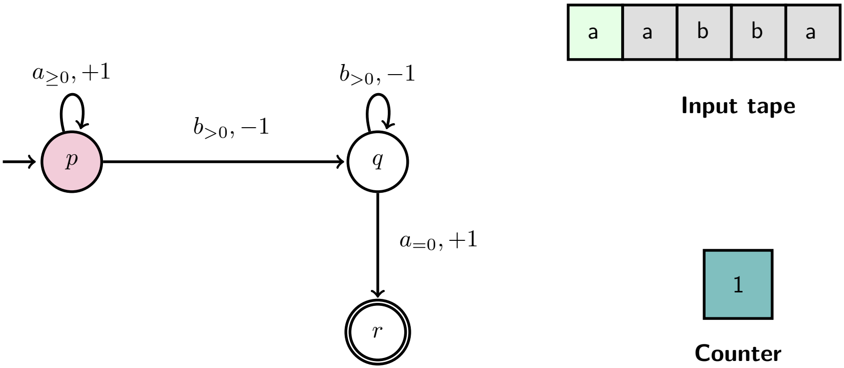

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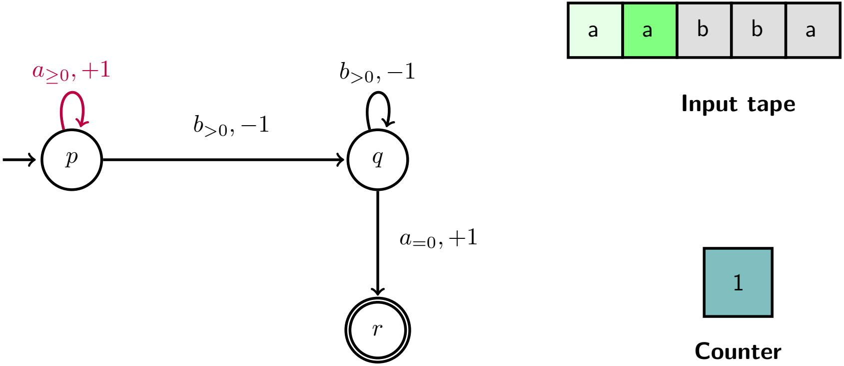

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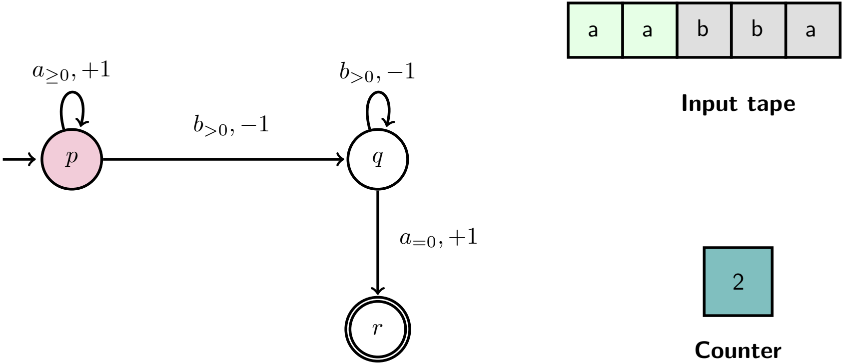

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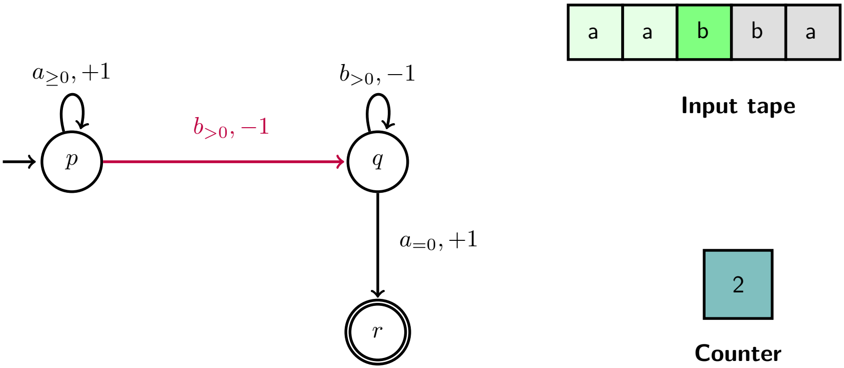

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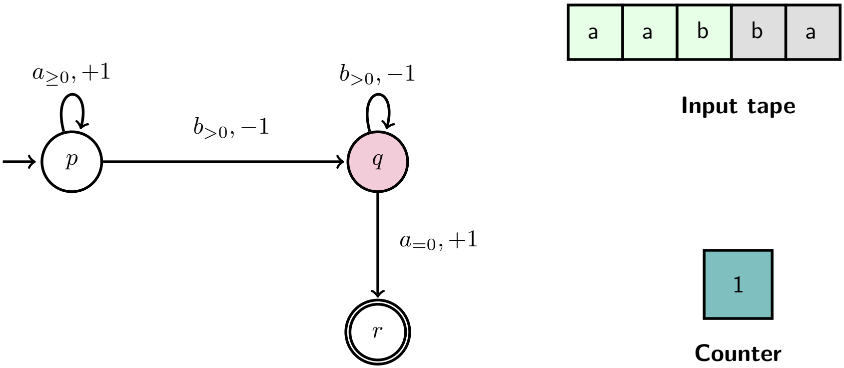

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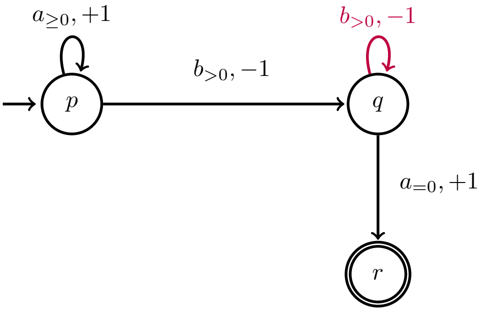

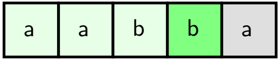

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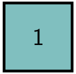

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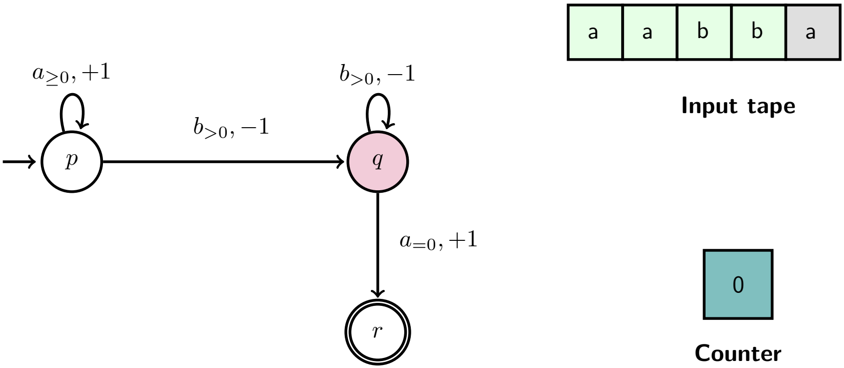

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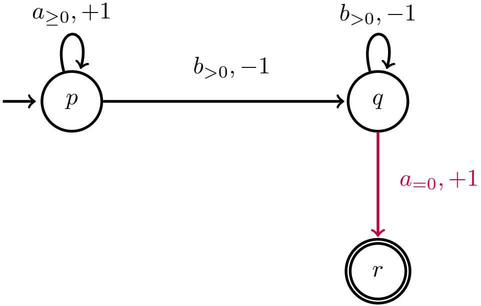

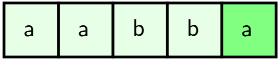

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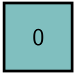

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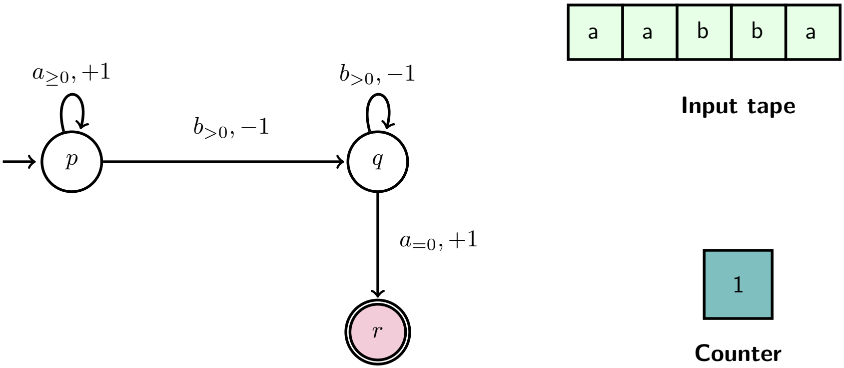

Motivation

Finite Automata $\subsetneq$ One-Counter Automata (OCA) $\subsetneq$ Pushdown Automata

- Modelling systems
  - Finite automata used extensively - eg. hardware verification.
  - Pushdown automata can model highly complex systems - eg. Softwares.
- Algorithmic complexity
  - Finite automata: Fast, mostly linear.
  - Pushdown automata: Hard, non-elementary to undecidable.
  - One-counter automata: Shows promise, some problems are theoretically good.
- Major challenges in OCA:
  - Equivalence - polynomial but $O(n^{20})$.
  - Active Learning - exponential.

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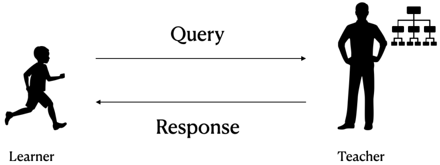
Active Learning Framework



- There are two parties: Learner and Teacher.
- The teacher knows the language of a doca $\mathcal{T}$.
- The learner wants to learn a doca $\mathcal{L}$ such that $\mathcal{T}$ and $\mathcal{L}$ accept the same language.
- The learner can ask the teacher questions about the language of $\mathcal{T}$.
- The teacher answers the questions.
- The learner use the answers to learn the doca $\mathcal{L}$.

Queries

Membership query

Learner: Is w in the language of $\mathcal{T}$?

Teacher: Yes or No.

Equivalence query

Learner: Is a doca $\mathcal{L}$ equivalent to $\mathcal{T}$?

Teacher: Yes or "No and a counter example w that distinguishes $\mathcal{L}$ and $\mathcal{T}$".

Minimal-equivalence query

Learner: Is a doca $\mathcal{L}$ equivalent to $\mathcal{T}$?

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Counter value query

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OL* - Active learning of doca[1]

Theorem (OL* in P)

- *Let teacher know a doca language.*
- *Let $\mathcal{T}$ be a minimal doca that accepts the language.*
- *Let $n = |\mathcal{T}|$ be the number of states in $\mathcal{T}$.*
- *The OL* algorithm learns a doca $\mathcal{L}$ that is equivalent to $\mathcal{T}$ in time polynomial in n, using membership and minimal-equivalence queries.*

[1] P. Mathew, V. Penelle, S. Learning deterministic one-counter automata in polynomial time, LICS 2025.

Literature review: Active learning of doca

- Angluin, 1987 → DFA – Membership, Equivalence
  P
- Fahmy & Roos, 1995 → DOCA^1 – Membership, Min. Equivalence
  $\mathrm{EXPTIME}$
- Neider & Löding, 2010 → VOCA^2 – Membership, Min. Equivalence
  $\mathrm{EXPTIME}$
- Bruyère et. al, 2022 → DOCA^1 – Membership, Min. Equivalence, Counter Value
  $\mathrm{EXPTIME}$
- Mathew et. al, 2025 → DOCA^1 – Membership, Min. Equivalence, Counter Value
  $\mathrm{EXPTIME}$, Polynomial queries

[1] realtime doca: strict subclass of doca,

[2] voca: visibly oca

The Configuration graph of a DOCA

Configuration graph of a doca

- Configuration: A pair (p, i) where p is a state and i is a counter value.
- Configuration graph:
  - States: all configurations (p, i).
  - Transitions: $(p, i) \xrightarrow{a} (q, j)$ if there is a transition from p to q on letter a and the counter value changes from i to j.
  - Final states: (p, i) where p is a final state.
  - Initial state: $(s, 0)$ where s is the start state.
- The configuration graph is infinite, if the oca is not a finite automata.

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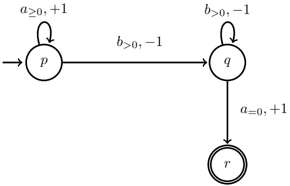
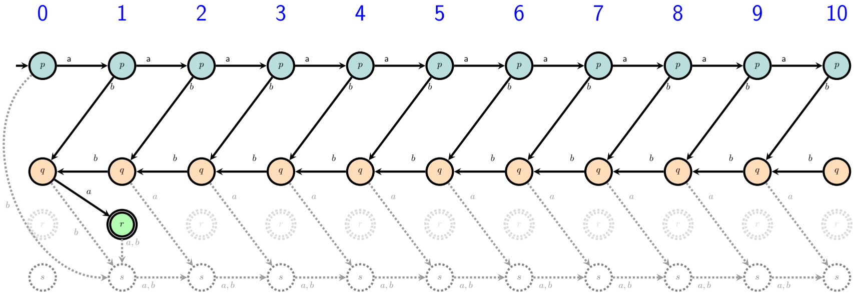
Configuration graph - Example





Configuration sequences

- Consider a doca with n states.
- Let p be a state, and integers $d \leq n^2$, and $i > n^3$.
- Consider sequence of configurations

$$(p, i),\ (p, i+d),\ (p, i+2d) \quad \ldots$$

- On taking letter a from these configurations, we get sequence

$$(q, i+c),\ (q, i+d+c),\ (q, i+2d+c) \quad \ldots$$

for some state q and $c \in \{-1, 0, +1\}$.

- For any word w, where $|w| \leq n^3$, there is a state r and counter j

$$(p, i),\ (p, i+d), \quad \ldots \quad \xrightarrow{w} \quad (r, j),\ (r, j+d), \quad \ldots$$

- What is unique about each sequence? state, counter value (mod d) pair

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Configuration sequences contd.

- Number of “non-intersecting” sequences are

$$|\text{ number of states }| \times d = nd \quad \leq n^3 \quad (\text{since } d \leq n^2)$$

- Let $(p, i + nd) \xrightarrow{v} (q, k)$ and the run do not touch a configuration with zero counter.
  - Then there is a w where $|w| \leq n^3$ such that

$$(p, i), (p, i + d), \quad \dots \quad \xrightarrow{w} \quad (q, j), (q, j + d), \quad \dots, (q, k = j + td), \dots$$

  - *Proof*: Let $(p, i + nd) \xrightarrow{v} (q, k)$ where $|v| > n^3$.
  - Hence, there is a $c \geq -n$ such that

$$(p, i + nd) \xrightarrow{u} (r, l) \xrightarrow{y} (r, l + cd) \xrightarrow{v} (q, k).$$

  - Then,

$$(p, i + cd) \xrightarrow{u} (r, l + cd) \xrightarrow{v} (q, k).$$

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  - Then there is a w where $|w| \leq n^3$ such that

$$(p, i), (p, i + d), \quad \ldots \quad \xrightarrow{\quad w \quad} \quad (q, j), (q, j + d), \quad \ldots, (q, k = j + td), \ldots$$

  - *Proof*: Let $(p, i + nd) \xrightarrow{v} (q, k)$ where $|v| > n^3$.
  - Hence, there is a $c \geq -n$ such that

$$(p, i + nd) \xrightarrow{\quad u \quad} (r, l) \xrightarrow{\quad y \quad} (r, l + cd) \xrightarrow{\quad v \quad} (q, k).$$

  - Then,

$$(p, i + cd) \xrightarrow{\quad u \quad} (r, l + cd) \xrightarrow{\quad v \quad} (q, k).$$

Configuration sequences contd.

- Number of "non-intersecting" sequences are

$$|\text{ number of states }| \times d = nd \quad \leq n^3 \quad (\text{since } d \leq n^2)$$

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Parallel Breadth First Search

- Consider the sequence

$$(p, i),\ (p, i + d),\ (p, i + 2d) \quad \ldots$$

- A parallel breadth first search (PBFS) will generate all sequences reachable without touching a zero configuration.
- The PBFS depth will be at most n^3.
- For a polynomially bounded sequence,

$$(p, i),\ (p, i + d),\ (p, i + 2d) \quad \ldots \quad (p, i + Kd)$$

PBFS will run in polynomial time.

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The OL* Algorithm

Assumptions on DOCA

- The teacher knows a language accepted by a doca.
  - $\mathcal{T}$ is a minimal doca equivalent to teacher's doca language.
  - We denote by $n = |\mathcal{T}|$, the number of states.
- To make the presentation simpler, we assume the following about $\mathcal{T}$:
  - There are no ε transitions.
  - In a transition, the counter is incremented or decremented at most by one.
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OL* - Algorithm

- OL* first assumes $n = 1$.
- It learns a doca $\mathcal{L}$ that is checked for equivalence with teacher.
- If teacher says $\mathcal{L}$ is not equivalent to $\mathcal{T}$, then n is incremented.
- Process continues with incremented n.
- If teacher says $\mathcal{L}$ is equivalent to $\mathcal{T}$, then OL* terminates.

- For proof of correctness, it suffices to show the following
  - For every n, OL* runs in time polynomial in n.
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OL*: Step 1. Learning Behaviour DFA

- Learner do not have access to the configuration graph of $\mathcal{T}$.
- $\mathcal{A}$ is a k-behaviour dfa if $\mathcal{A}$ is k-equivalent to $\mathcal{T}$. That is,

$$w \text{ is accepted by } \mathcal{A} \quad \text{iff} \quad w \text{ is accepted by } \mathcal{T}, \quad \text{for all } |w| \leq k.$$

- Angluin's L^* algorithm can learn a k-behaviour dfa in time polynomial in k and n.
  - This is where minimal-equivalence is used.
- Step 1. of learner is to learn a $poly(n)$-behaviour dfa.
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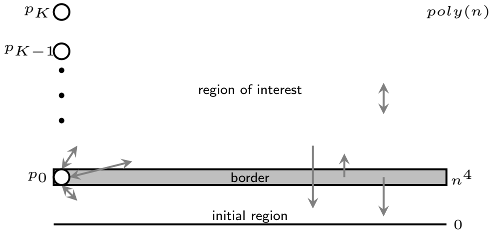
OL*: Step 2. Partitioning the behaviour DFA

The dfa $\mathcal{A}$ is partitioned into:

- Initial region: States reachable by words of length $< n^4$.
- Border region: States reachable by words of length n^4 but not less.
- Region of interest: Remaining states.



- A path from initial region to region of interest should traverse via some border state.
- Partial OCA construction
  - Pick a border state p_0.
  - DFA $\mathcal{A}_{p_0}$: Remove all states other than p_0 from border.
  - Learner constructs a partial OCA, $\mathcal{L}_{p_0}$ that is $poly(n)$-equivalent to $\mathcal{A}_{p_0}$.

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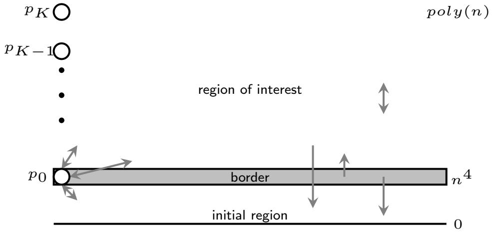

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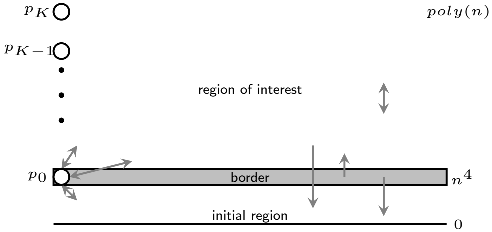

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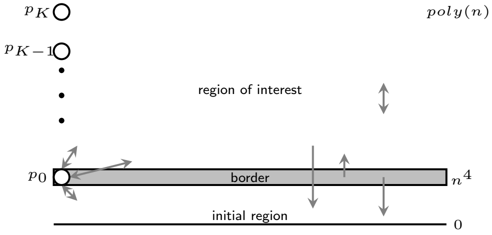

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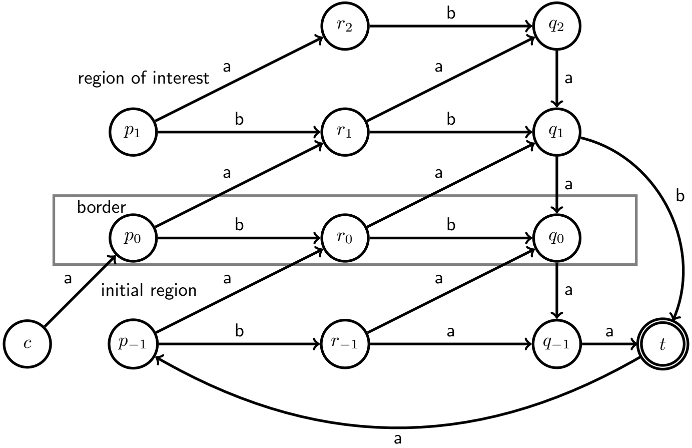
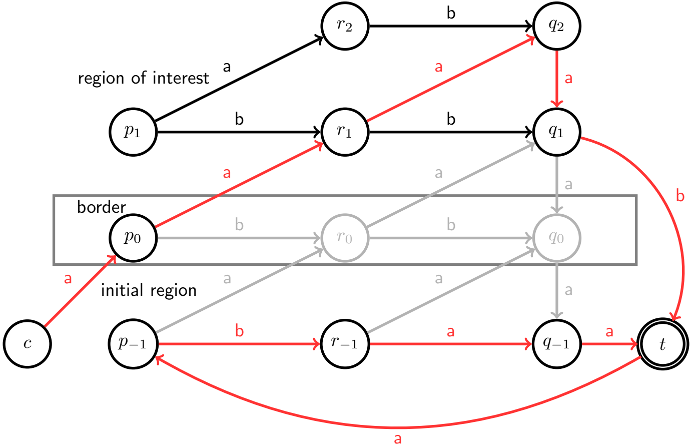


r_2
b
q_2
a
region of interest
a
a
p_1
b
r_1
b
q_1
a
a
a
b
border
p_0
b
r_0
b
q_0
a
initial region
a
a
a
c
p_{-1}
b
r_{-1}
a
q_{-1}
a
t
a


region of interest
border
initial region
r_2
q_2
p_1
r_1
q_1
p_0
r_0
q_0
c
p_{-1}
r_{-1}
q_{-1}
t
a
b

OL*: Step 3. Finding a winning sequence

Definition

$$w_0, w_1, w_2, \quad \ldots, \quad w_K$$

is a winning sequence if the run of these words on $\mathcal{T}$ reach configurations

$$(p, i),\ (p, i+d),\ (p, i+2d), \quad \ldots, \quad (p, i+Kd)$$

respectively, for some state p, and $d \leq n^2$ and $i > n^3$.

Lemma (Winning sequence lemma)

For any state p_0 in behaviour dfa $\mathcal{A}$, a winning sequence

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OL*: Step 4. Parallel BFS on $\mathcal{A}$

- Consider a *winning sequence*

$$w_0, w_1, w_2, \quad \dots, \quad w_K$$

- Run these words on the behaviour dfa. We reach state sequence

$$p_0, p_1, p_2, \quad \dots, \quad p_K$$

- Run parallel BFS (depth at most n^3) from this sequence.
  - All distinct sequences identified.
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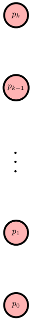
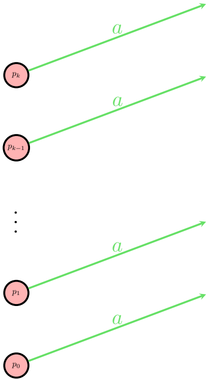
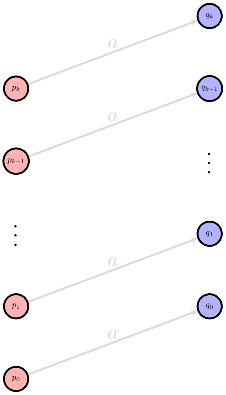
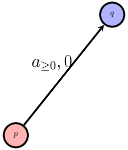
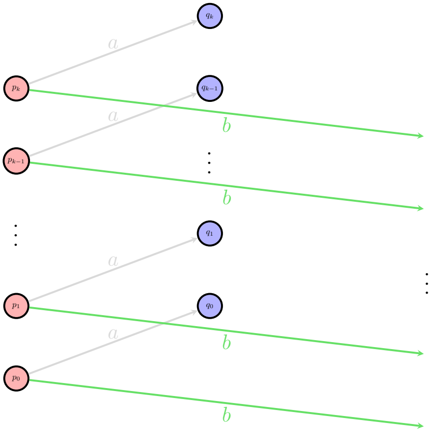
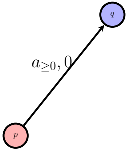

p_k
p_{k-1}
p_1
p_0


p_k
a
p_{k-1}
a
p_1
a
p_0
a


q_k
a
p_k
q_{k-1}
a
p_{k-1}
q_1
a
p_1
q_0
a
p_0


q
$a_{\geq 0}, 0$
p


q_k
a
p_k
q_{k-1}
a
b
p_{k-1}
b
q_1
a
q_0
p_1
a
b
p_0
b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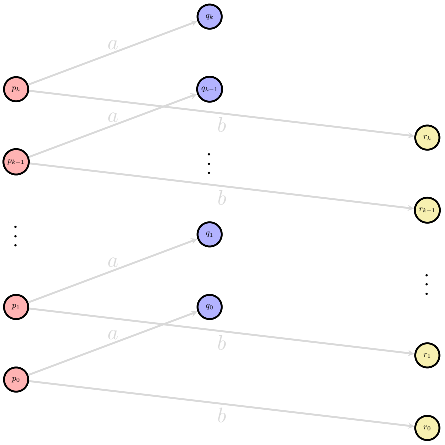
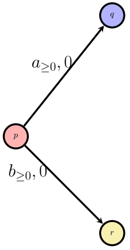
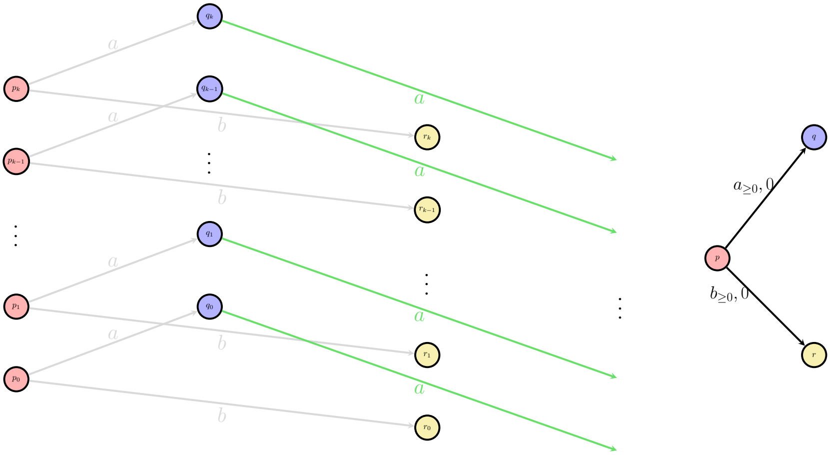
$a_{\geq 0}, 0$
p


q_k
q_{k-1}
q_1
q_0
p_k
p_{k-1}
p_1
p_0
r_k
r_{k-1}
r_1
r_0
a
b


p
q
r
$a_{\geq 0}, 0$
$b_{\geq 0}, 0$


q_k
q_{k-1}
q_1
q_0
p_k
p_{k-1}
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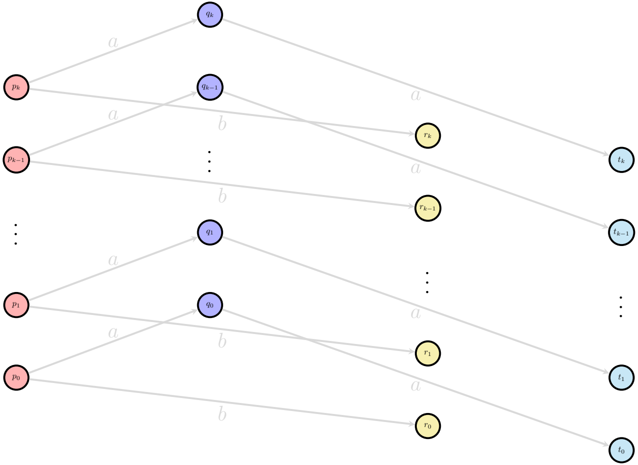
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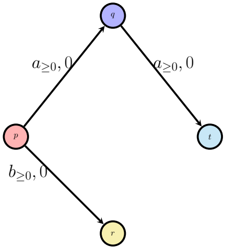
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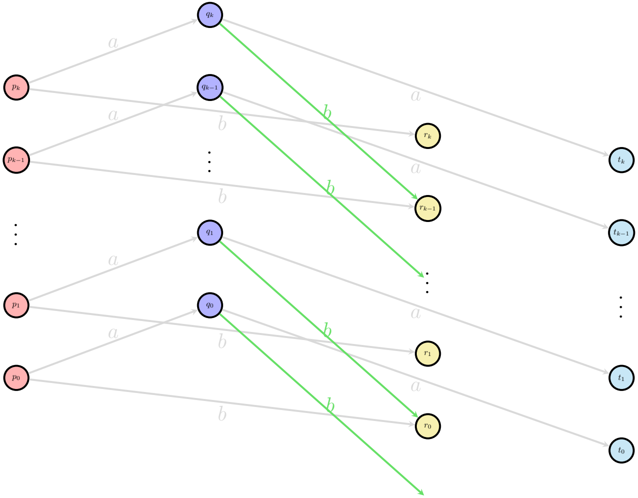
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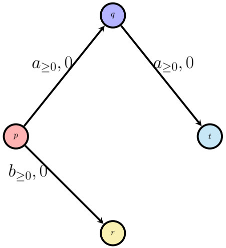
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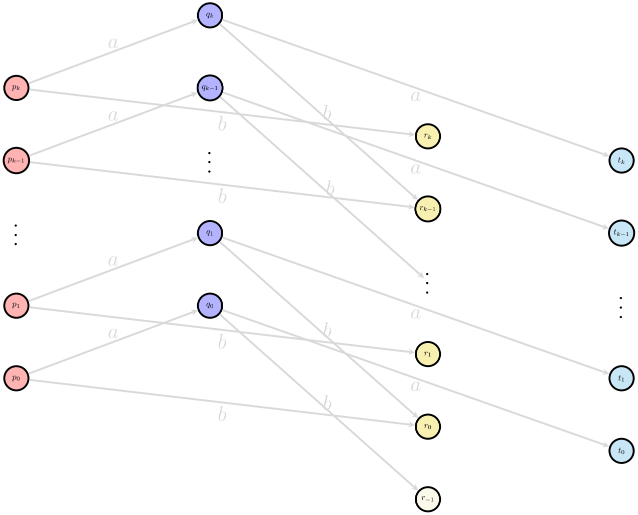
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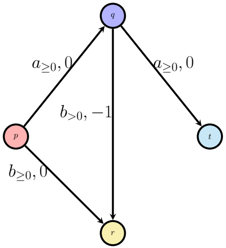
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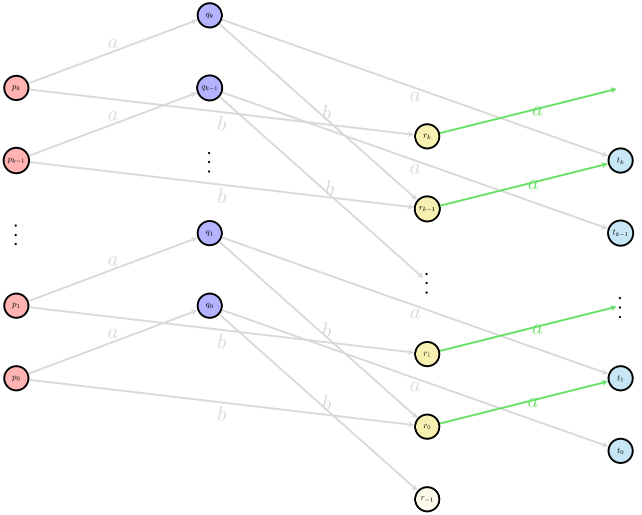
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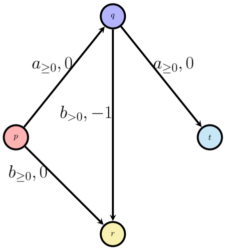
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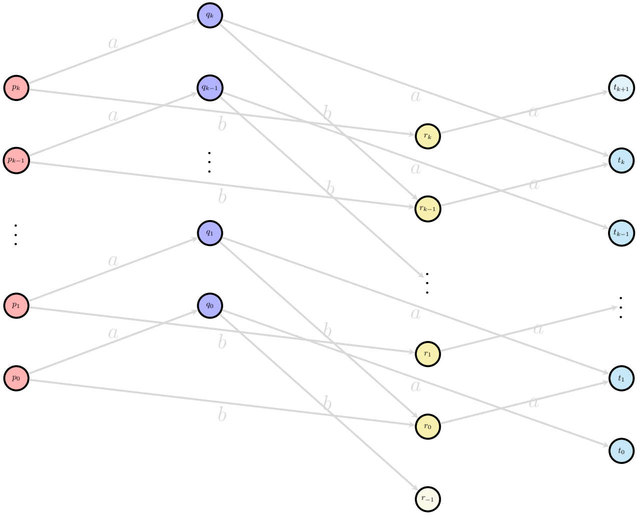
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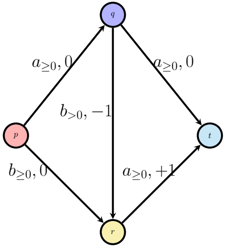
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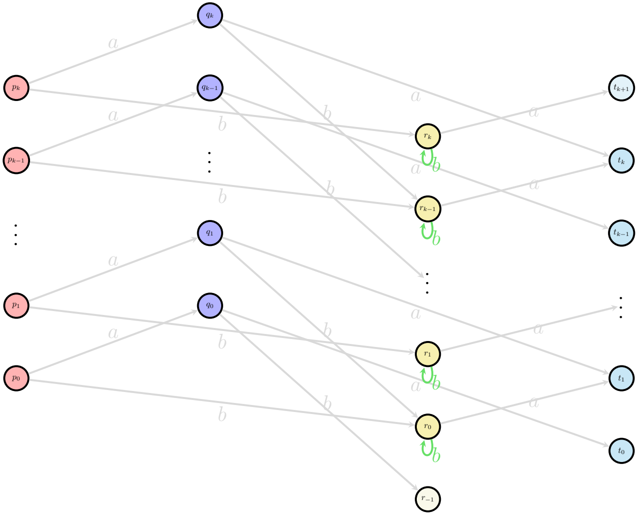
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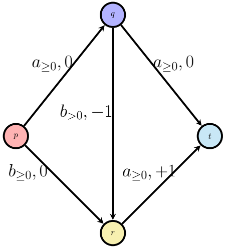
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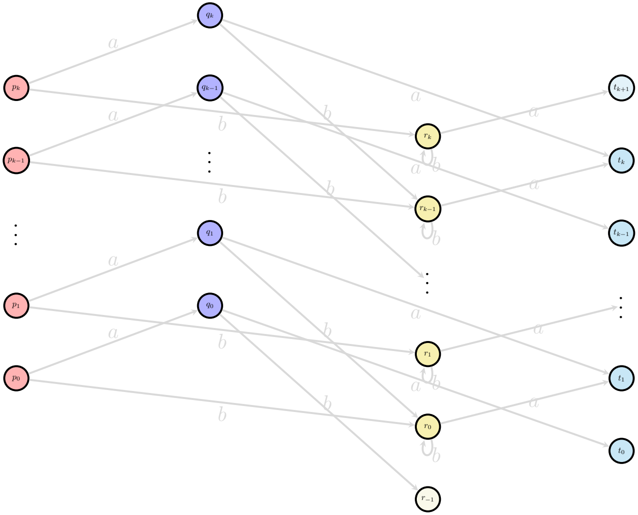
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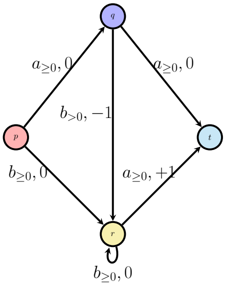
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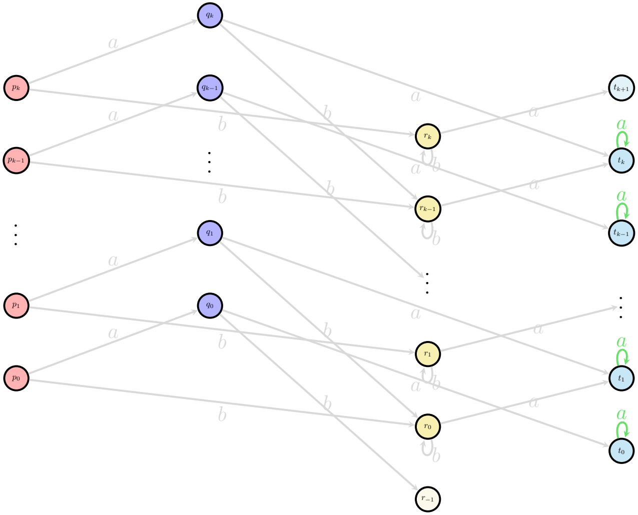
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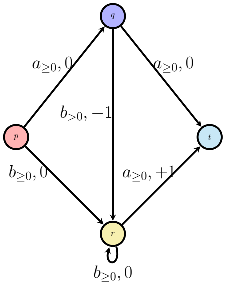
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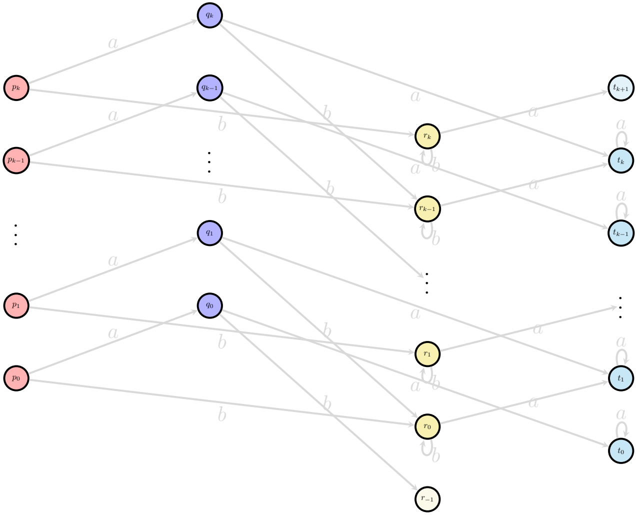
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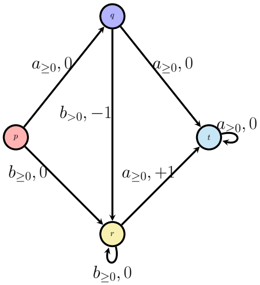
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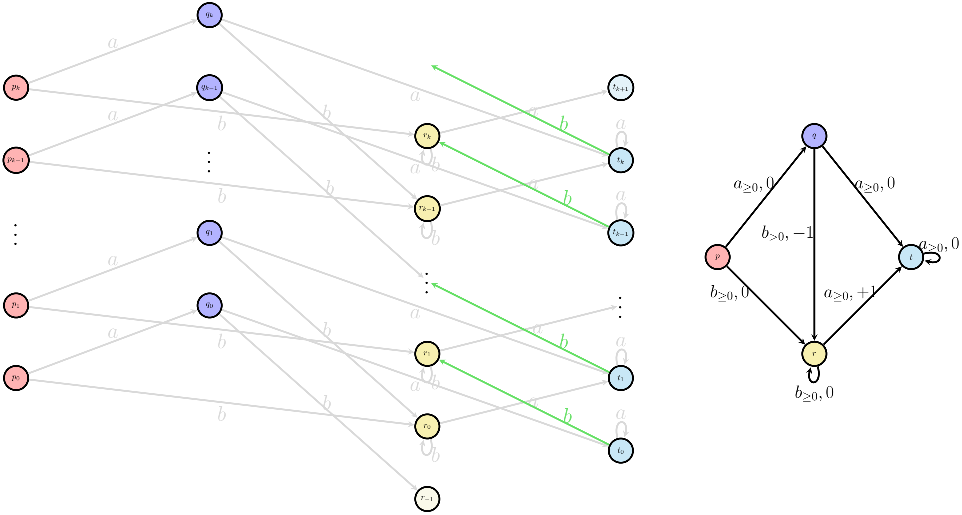
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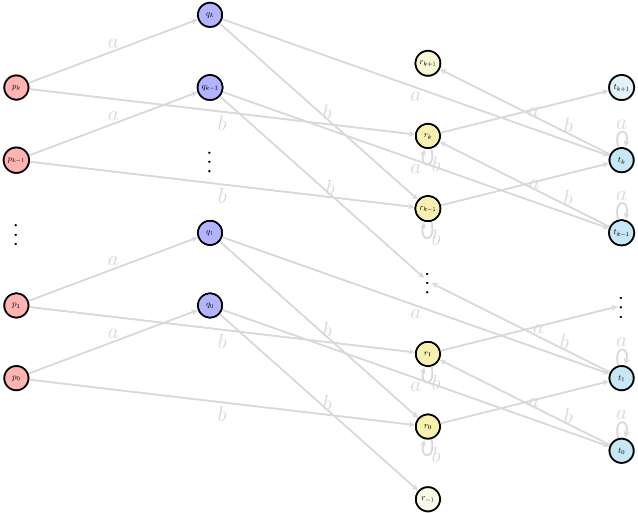
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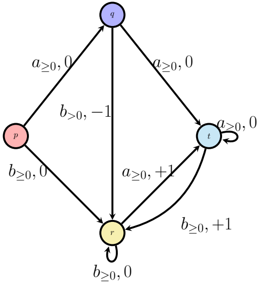
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OL*: Step 5. Constructing $\mathcal{L}_{p_0}$

- The states of the partial OCA are the sequences.
  - Let (p_0, p_K) be one sequence - call this Red sequence.
  - Let (q_0, q_K) be another sequence - call this $Blue$ sequence.
  - Then Red and $Blue$ are states of partial OCA.
- Transitions:
  - If $(p_0, p_K) \xrightarrow{a} (q_0, q_K)$, add transition $Red \xrightarrow{a \geq 0, 0} Blue$.
  - If $(p_0, p_{K-1}) \xrightarrow{a} (q_1, q_K)$, add transition $Red \xrightarrow{a \geq 0, +1} Blue$.[1]
  - If $(p_1, p_K) \xrightarrow{a} (q_0, q_{K-1})$, add transition $Red \xrightarrow{a > 0, -1} Blue$.[1]
- Hence:
  - $p_i \xrightarrow{a} q_j$ if and only if $(Red, i) \xrightarrow{a} (Blue, j)$.
  - If $(Red, i) \xrightarrow{w} (Blue, j)$, then $p_i \xrightarrow{w} q_j$.
  - If $p_i \xrightarrow{a} r_k \xrightarrow{b} s_l \ldots \xrightarrow{a} q_j$, then $(Red, i) \xrightarrow{ab\ldots a} (Blue, j)$.

[1] $p_n \xrightarrow{a} q_\ell$ for $0 \leq \ell \leq 2n$ is possible.

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Patch work - Missing states

- The parallel BFS colors "most" of the reachable states from border state p.
- However, upto n^3 number of states are not colored (called Neg states)
  - eg. the state r_{-1} in the example, and some states reachable from r_{-1}.
- The Neg states are added to the partial OCA.
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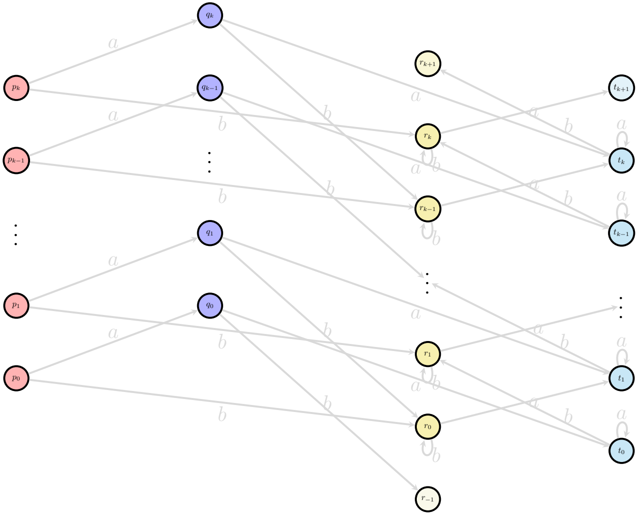
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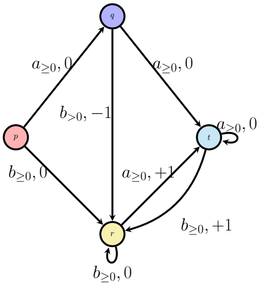
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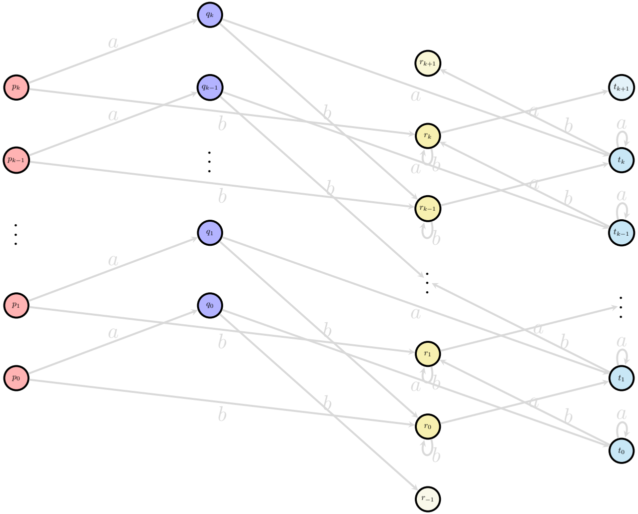
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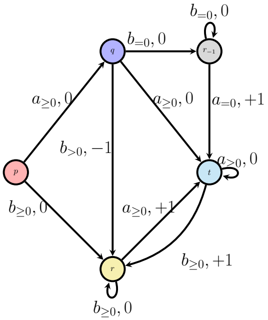
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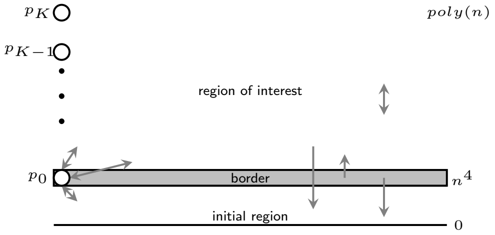
Adding initial region

- The initial-region is added to the partial OCA.
- Like Neg states, initial region do not increment or decrement counter.



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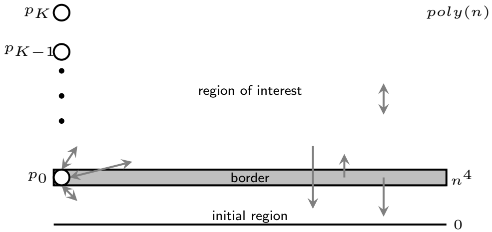

This concludes the construction of partial OCA $\mathcal{L}_{p_0}$.

Partial OCA $\mathcal{L}_{p_0}$ - Properties

Theorem

Let w be a word of length $\leq poly(n)$. Then one of the following holds:

- *Either*

$$\mathcal{L}_p \text{ accepts } w \quad \text{iff} \quad \mathcal{A} \text{ accepts } w$$

- *or there is a prefix u such that the run of u on $\mathcal{A}$ reaches a border state $q \neq p$.*

OL*: Step 6. Constructing $\mathcal{L}$

- The final OCA $\mathcal{L}$ is union of partial OCAs $\mathcal{L}_p, \mathcal{L}_q, \ldots, \mathcal{L}_r$ where $border = \{p, q, \ldots, r\}$
- From, construction of $\mathcal{L}_p$: for any word w where $|w| \leq poly(n)$:

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- Hence, for any word w where $|w| \leq poly(n)$:

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- [Böhm et. al., 2013] There is a polynomial $poly(n)$ such that if $\mathcal{L}$ is $poly(n)$-equivalent to $\mathcal{T}$, then $\mathcal{L}$ is equivalent to $\mathcal{T}$.
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$$\mathcal{L} \quad \text{is equivalent to} \quad \mathcal{T}$$

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OL*: Step 6. Constructing $\mathcal{L}$

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Summary of OL*

- Construct $poly(n)$-behaviour DFA using L* algorithm.
- Partition the behaviour DFA into *initial region*, *border*, and *region of interest*.
- For each border state:
  - Generate a winning sequence of words: $w_0, w_1, \ldots, w_K$.
  - Run these words on the DFA to get sequence of states: $p_0, p_1, \ldots, p_K$.
  - Run parallel BFS from this sequence.
  - All reachable sequences of parallel BFS form states of partial OCA.
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Proof of winning sequence lemma

Winning sequence lemma

Definition

$$w_0, w_1, w_2, \quad \ldots, \quad w_K$$

is a winning sequence if the run of these words on $\mathcal{T}$ reach configurations

$$(p, i),\ (p, i+d),\ (p, i+2d), \quad \ldots, \quad (p, i+Kd)$$

respectively, for some state p, and $d \leq n^2$ and $i > n^3$.

Lemma (Winning sequence lemma)

For any state p_0 in behaviour dfa $\mathcal{A}$, a winning sequence

$$w_0, w_1, w_2, \quad \ldots, \quad w_K$$

can be found in polynomial time, such that the run of w_0 on $\mathcal{A}$ reaches state p_0.

Proof of winning sequence lemma

- Let $(s, 0)$ be the start configuration of a doca.
- For a configuration (p, i), we say

$$w = llex(p, i)$$

  if w is the lexicographically minimal word that takes $(s, 0)$ to (p, i).
- That is,

$$(s, 0) \xrightarrow{w} (p, i), \quad \text{and}$$

$$(s, 0) \xrightarrow{u} (p, i) \implies (|w|, w) \leq (|u|, u), \quad \text{for all } u.$$

Proof of winning sequence lemma contd.

Lemma

- *Let (p, i) be a configuration where $i > n^3$. Then,*

$$llex(p, i) = xy^r z, \quad \textit{where } |x|, |y|, |z| \leq n^3, \textit{ and } y \textit{ increases counter by } \leq n^2.$$

- *Furthermore,*

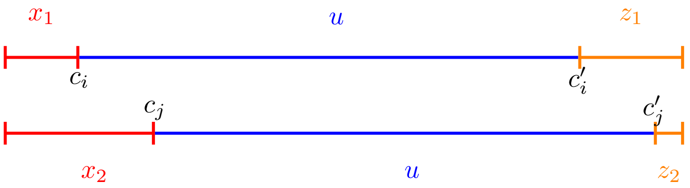
$$(s, 0) \xrightarrow{xy^{r+j}z} (p, i + jd), \quad \textit{for all } j \geq 0, \textit{ and } d \leq n^2.$$

Proof:

- Let $w = llex(p, i)$.
- Let c_i be the last configuration where counter value i is seen for the last time.

Repeating Factor

- c_i is the configuration where counter value i is seen for the last time.
- there are c_i and c_j with same state and c'_i and c'_j with same state.



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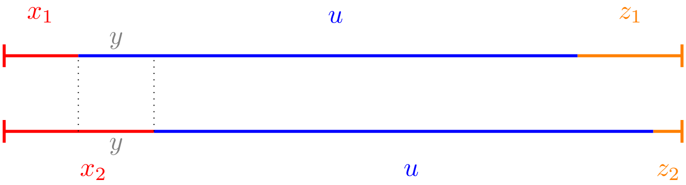

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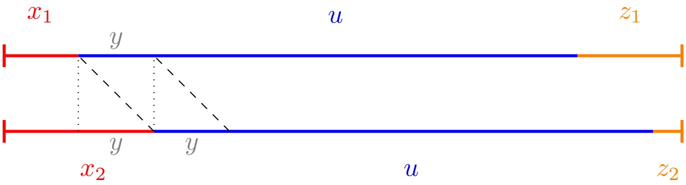

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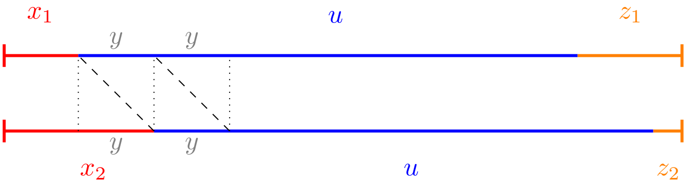

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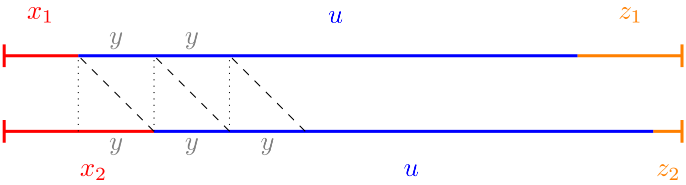

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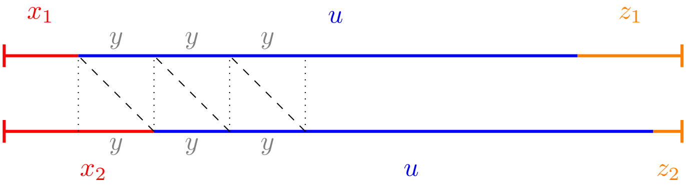

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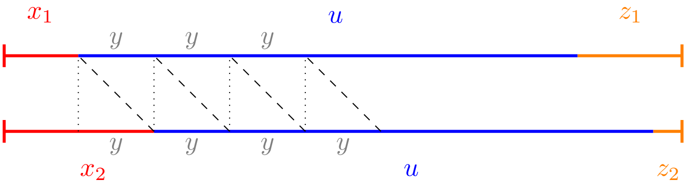

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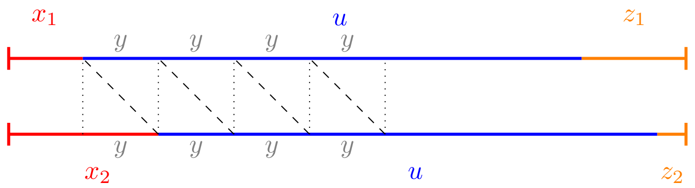

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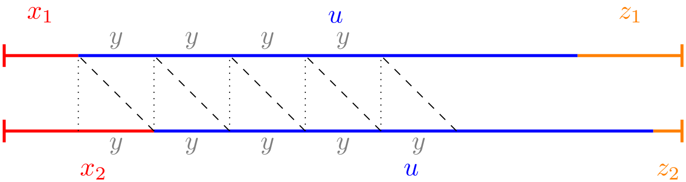

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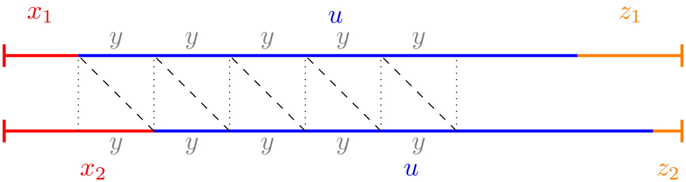

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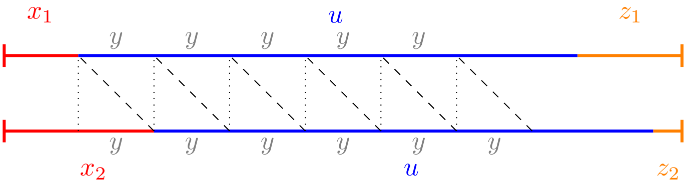

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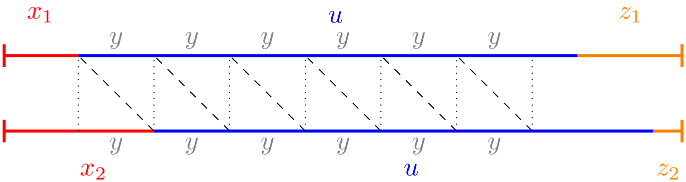

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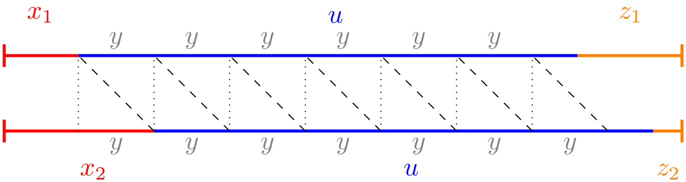

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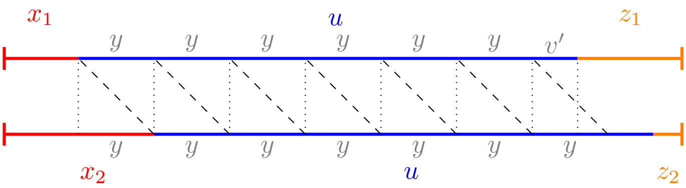

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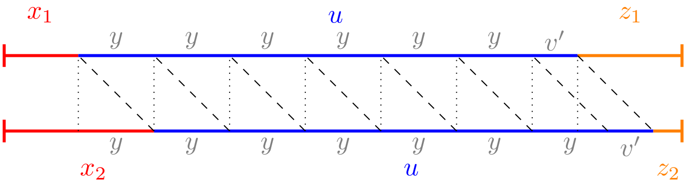

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Theorem

OL learns a doca equivalent to the teacher's doca using membership and minimal-equivalence queries, and in time polynomial in the size of a smallest doca recognizing the language.*

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Future work

- Replacing minimal-equivalence with equivalence query.
- Practical OL* algorithm.
- Improving running time of equivalence.
- Learning weighted models (like visibly OCA).

Thank You!