SAT-Based Invariant Inference and Its Relation to Concept Learning

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erc Supervised Verification of Infinite-State Systems

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Neil Immerman

UNIVERSITY OF MASSACHUSETTS AMHERST 1863

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CERTORA

James R. Wilcox

UNIVERSITY of WASHINGTON

SAT-Based Invariant Inference

- predicate abstraction
  [CAV’97, POPL’02]
- symbolic abstraction
  [VMCAI’04,’16]
- interpolation
  [CAV’03, TACAS’06]
- IC3/PDR
  [VMCAI’11, FMCAD’11]
- abduction
  [OOPSLA’13]
- SyGuS
  [FMCAD’13,…]
- ICE learning
  [CAV’14, POPL’15]
- …

Why do they succeed?

Why do they fail?

(How can we make them better?)

Goal

Understand SAT-based invariant inference
from the perspective of exact learning with queries

[POPL’20] Complexity and information in invariant inference. Feldman, Immerman, Sagiv, Shoham
[POPL’21] Learning the boundary of inductive invariants. Feldman, Sagiv, Shoham, Wilcox
[POPL’22] Property-directed reachability as abstract interpretation in the monotone theory. Feldman, Sagiv, Shoham, Wilcox
[SAS’22] Invariant Inference With Provable Complexity From the Monotone Theory. Feldman, Shoham

Safety of Transition Systems

Safety: no bad state is reachable from the initial states

Init:

$$(x_1, \ldots, x_n) \coloneqq 0 \ldots 0$$

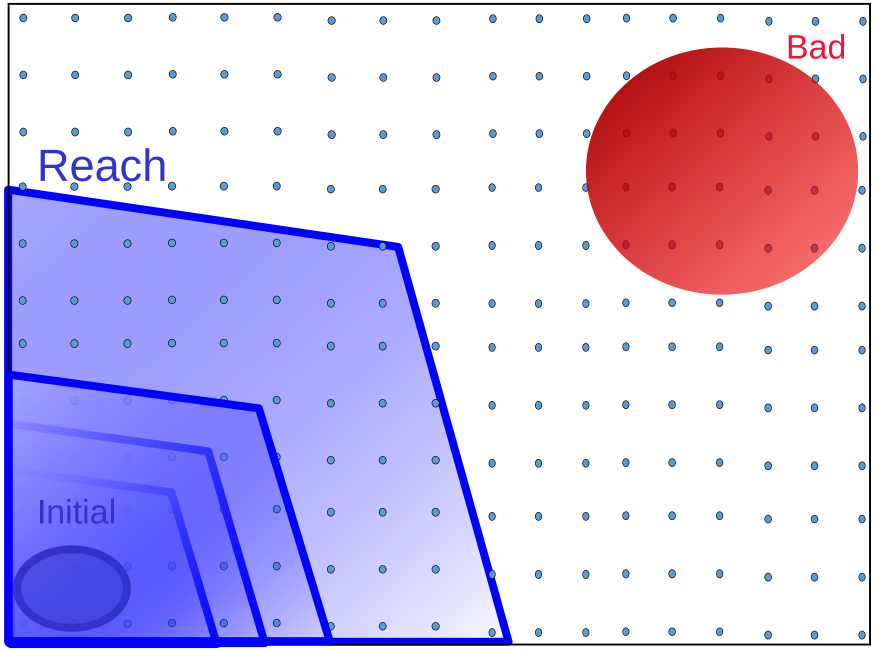
Bad:

$$(x_1, \ldots, x_n) = 1 \ldots 1$$

δ:

$$y_1, \ldots, y_n \coloneqq *$$

$$x_1, \ldots, x_n \coloneqq (x_1, \ldots, x_n) + 2 \cdot (y_1, \ldots, y_n) \pmod{2^n}$$



Inductive Invariants

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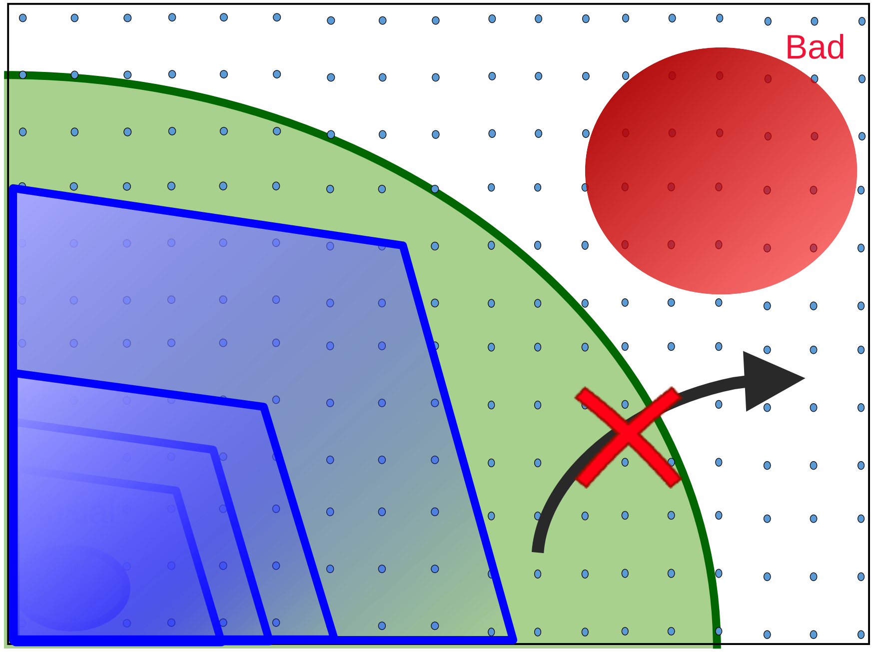

Initiation: $\text{Init} \subseteq I$

Safety: $I \cap \text{Bad} = \emptyset$

Consecution: $\{I\}\ \delta\ \{I\}$

Inductive Invariants

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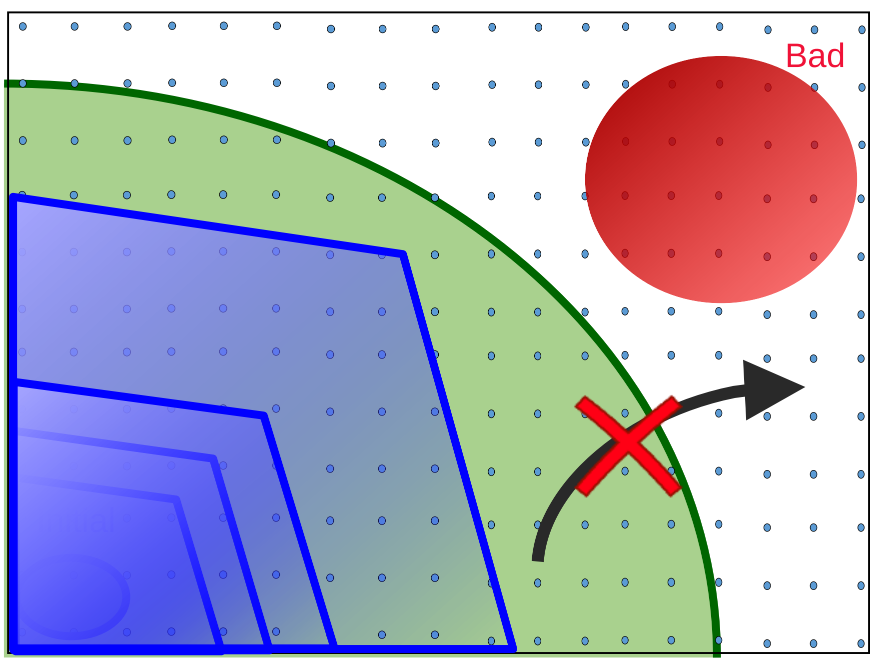
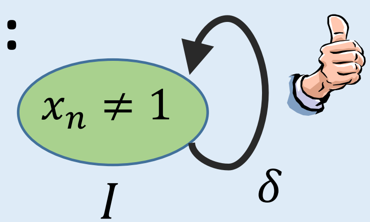


I: $x_n \neq 1$

Inductive:



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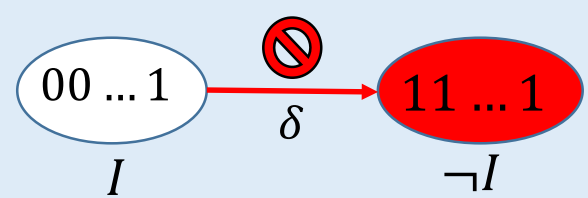
Not inductive:



Invariant Inference

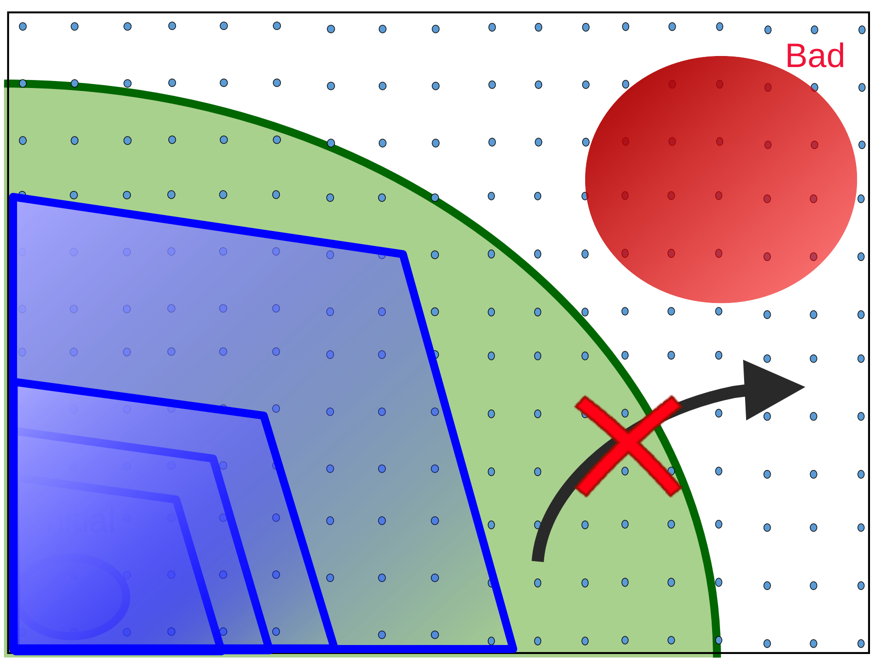
Safety: no bad state is reachable from the initial states

Goal: Find inductive invariants automatically

Bad:

$(x_1, \dots, x_n) = 1 \dots 1$

$2 \cdot (y_1, \dots, y_n) \pmod{2^n}$



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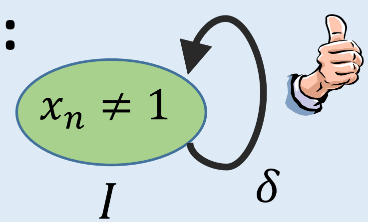

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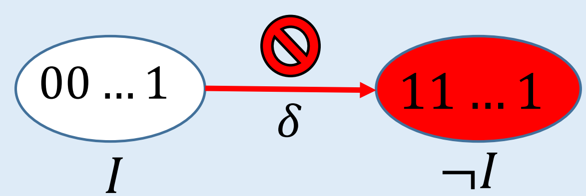
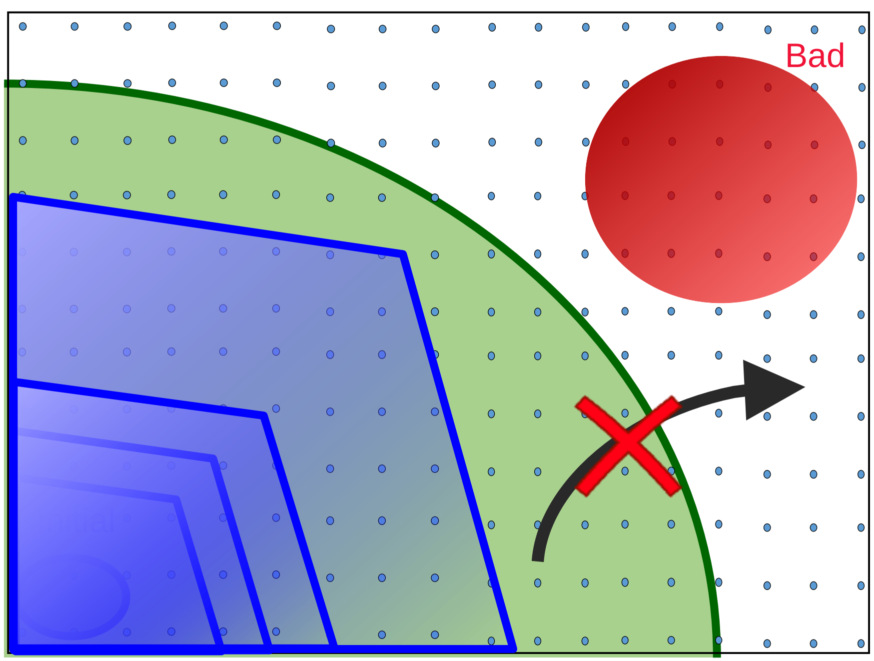


SAT-based Invariant Inference

Goal: Find inductive invariants automatically

Means: Employ a SAT solver



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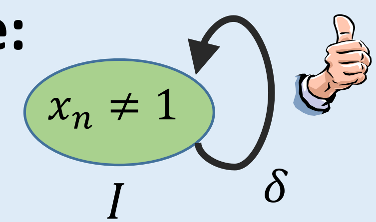

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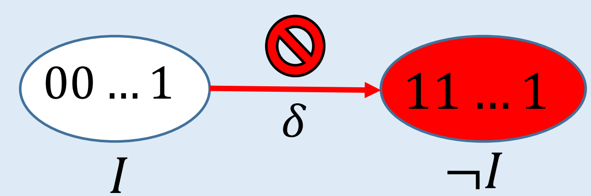

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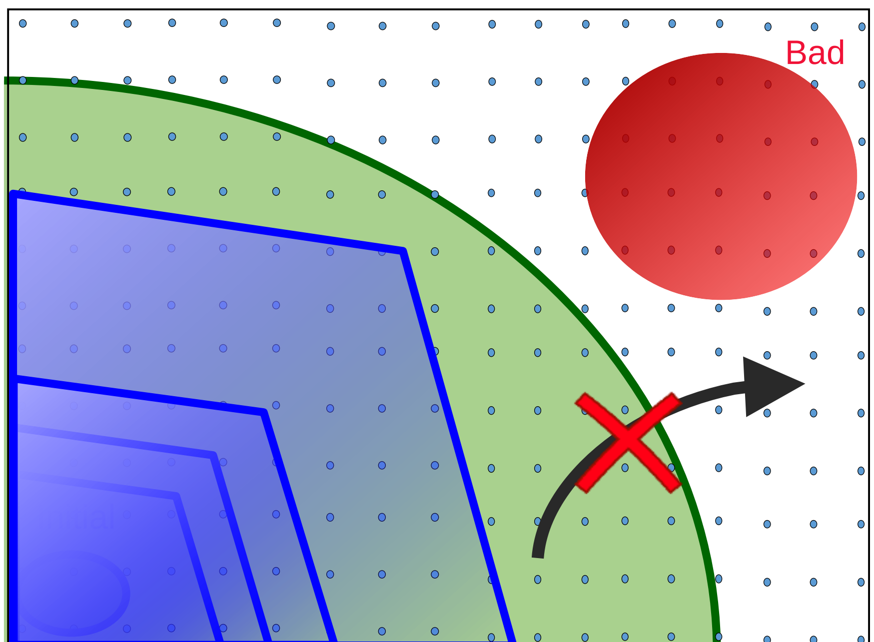


Init, Bad: formulas over V

$\boldsymbol{\delta}$: formula over V, V'

SAT query Examples:

Initiation: **Init** $\wedge \neg \boldsymbol{I}$ unsat?

Safety: $\boldsymbol{I} \wedge$ **Bad** unsat?

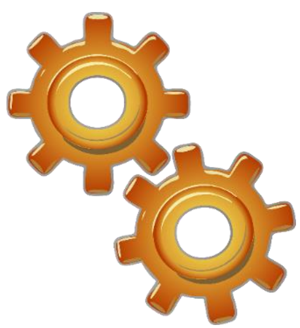
Cons.: $\boldsymbol{I} \wedge \boldsymbol{\delta} \wedge \neg \boldsymbol{I'}$ unsat?

* $I' = I[V \mapsto V']$

Exact Concept Learning with Equivalence & Membership Queries

Goal: learn an unknown concept φ

learning algorithm



is it ψ_1?

✓ / ✗+counterexample

is it ψ_2?

✓ / ✗+counterexample

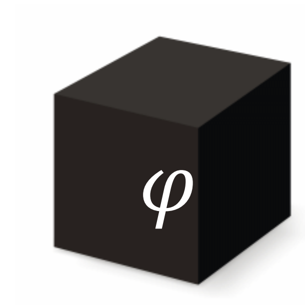
does $\sigma_3 \vDash$?

✓ / ✗

...

Membership Equivalence

oracle

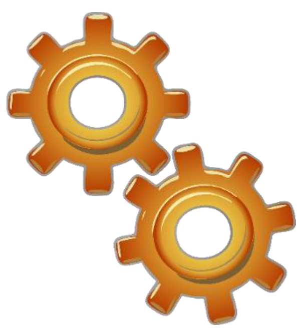


[ML'87] Queries and Concept Learning. Angluin

SAT-Based Invariant Inference as Inference with Queries

Goal: infer an unknown inductive invariant I

learning algorithm ↦ inference algorithm



Q →

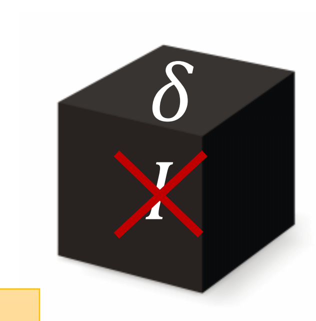
A ←

Q →

A ←

Q

oracle ↦ SAT-solver

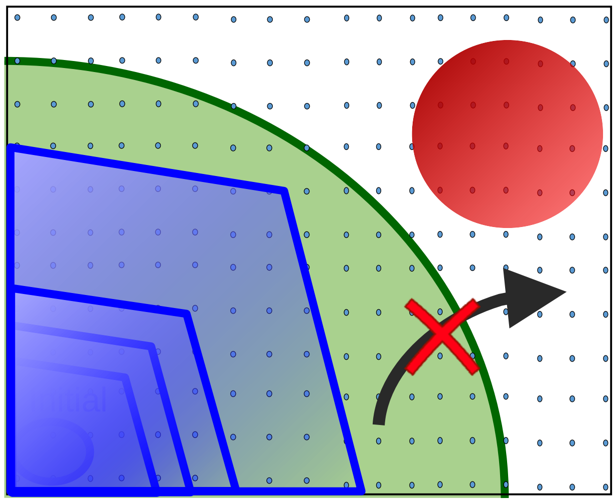


Which SAT queries?

Algorithms cannot access the transition relation directly, only through SAT queries

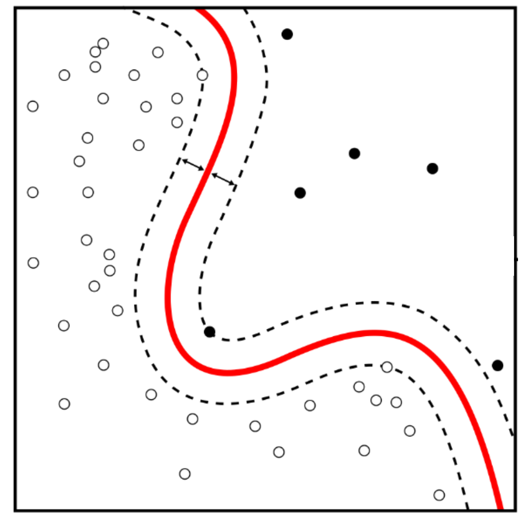
This Talk

Invariant Inference



vs.

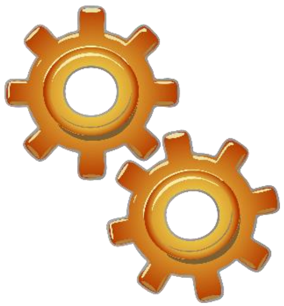
Exact Concept Learning



- Query-based learning models for invariant inference
- Complexity lower and upper bounds for each model
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from concept learning algorithms

Inductiveness-Query Model

inference algorithm



α_1 inductive? →

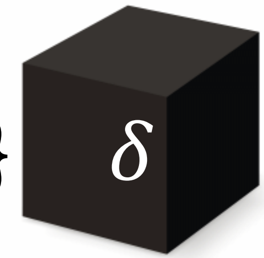
← ✓ / ✗+counterexample

…

α_m inductive? →

← ✓ / ✗+counterexample

inductiveness-query oracle

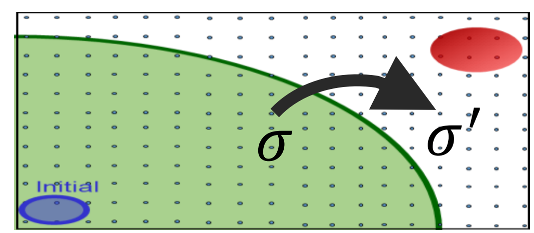
$\{\alpha_i\}$  $\{\alpha_i\}$

$\boldsymbol{\alpha_i \wedge \delta \wedge \neg\alpha_i'}$ unsat?

Cex to Induction (CTI):
Transition (σ, σ') of δ s.t.
$\sigma \vDash \alpha_i, \quad \sigma' \vDash \neg\alpha_i$

ICE framework - Learn from examples:

Positive : $\sigma \vDash I$ (e.g., initial)
Negative: $\sigma \nvDash I$ (e.g., bad)
Implication: $\sigma \vDash I$ implies $\sigma' \vDash I$ (CTI)

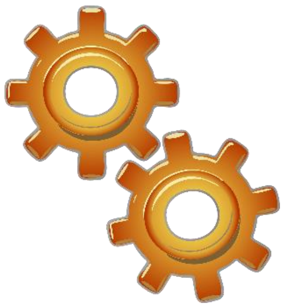


* $\boldsymbol{\alpha_i'}=\boldsymbol{\alpha_i}[V \mapsto V']$

[CAV’14] ICE: A Robust Framework for Learning Invariants. Garg, Löding, Madhusudan, Neider

Inductiveness-Query Model

inference algorithm



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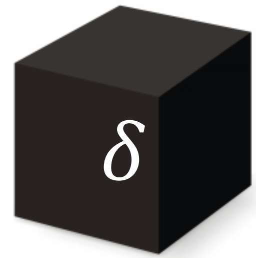
$\{\alpha_i\}$ 

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Is it sufficient to capture existing SAT-based algorithms?

* $\alpha_i' = \alpha_i[V \mapsto V']$

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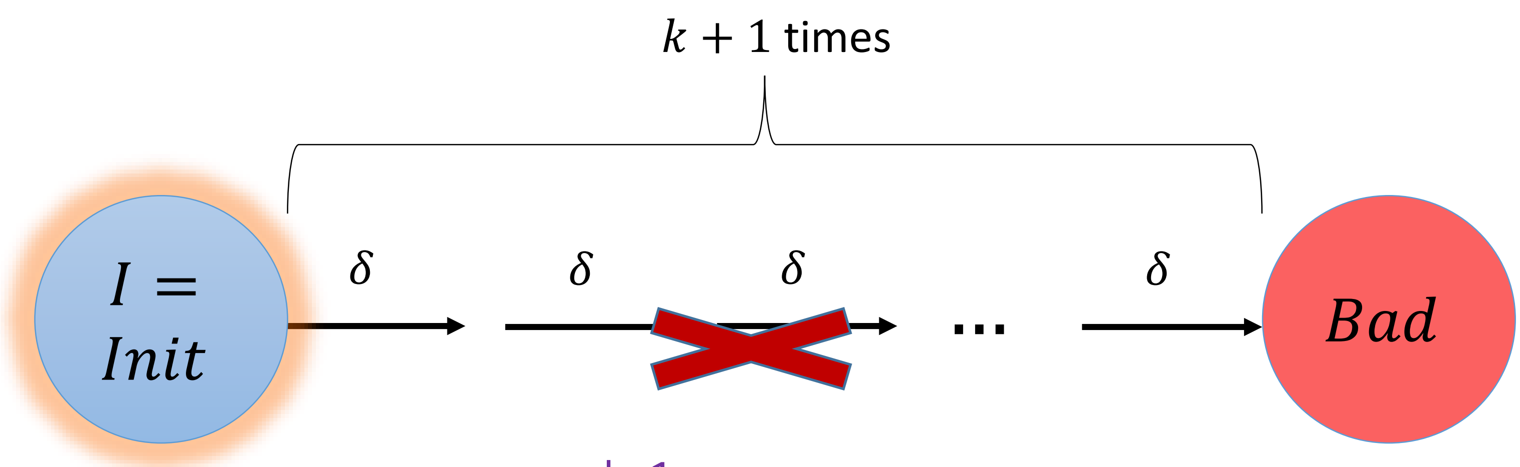
Interpolation-Based Inference

$I = Init$ **Inductive ?**

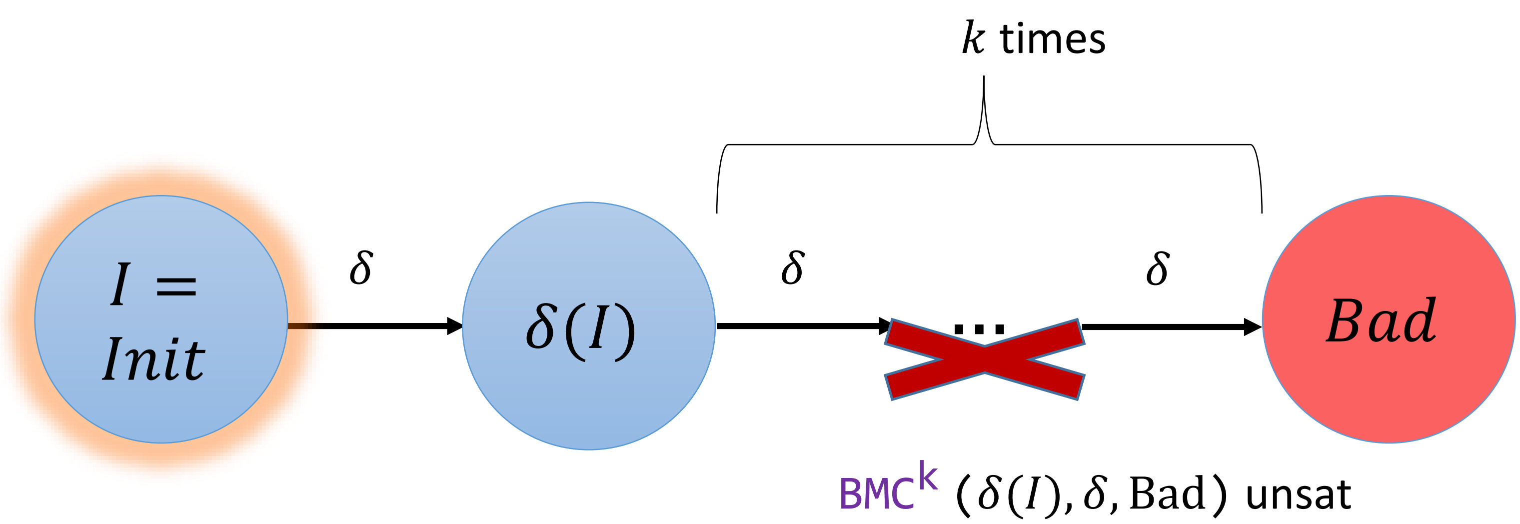


BMC^{k+1} $(I, \delta, \mathrm{Bad})$ unsat

[CAV'03] Interpolation and SAT-Based Model Checking, McMillan

Interpolation-Based Inference

$I = Init$



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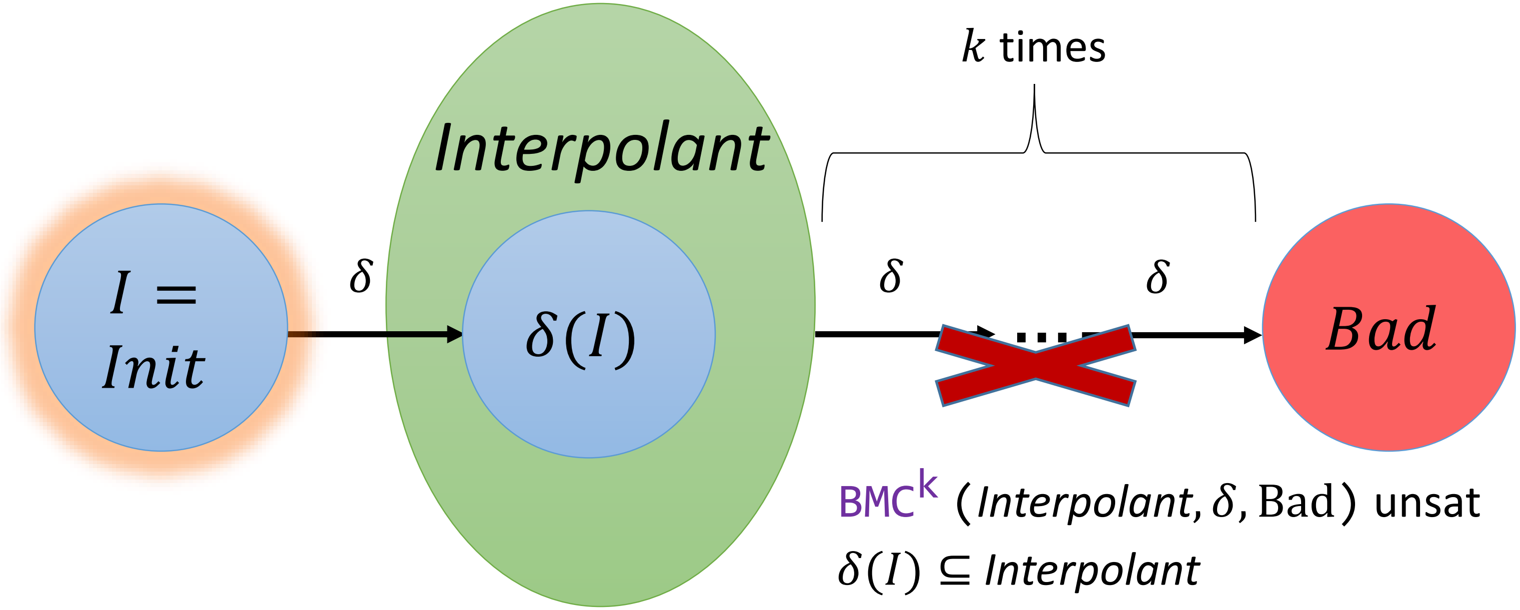


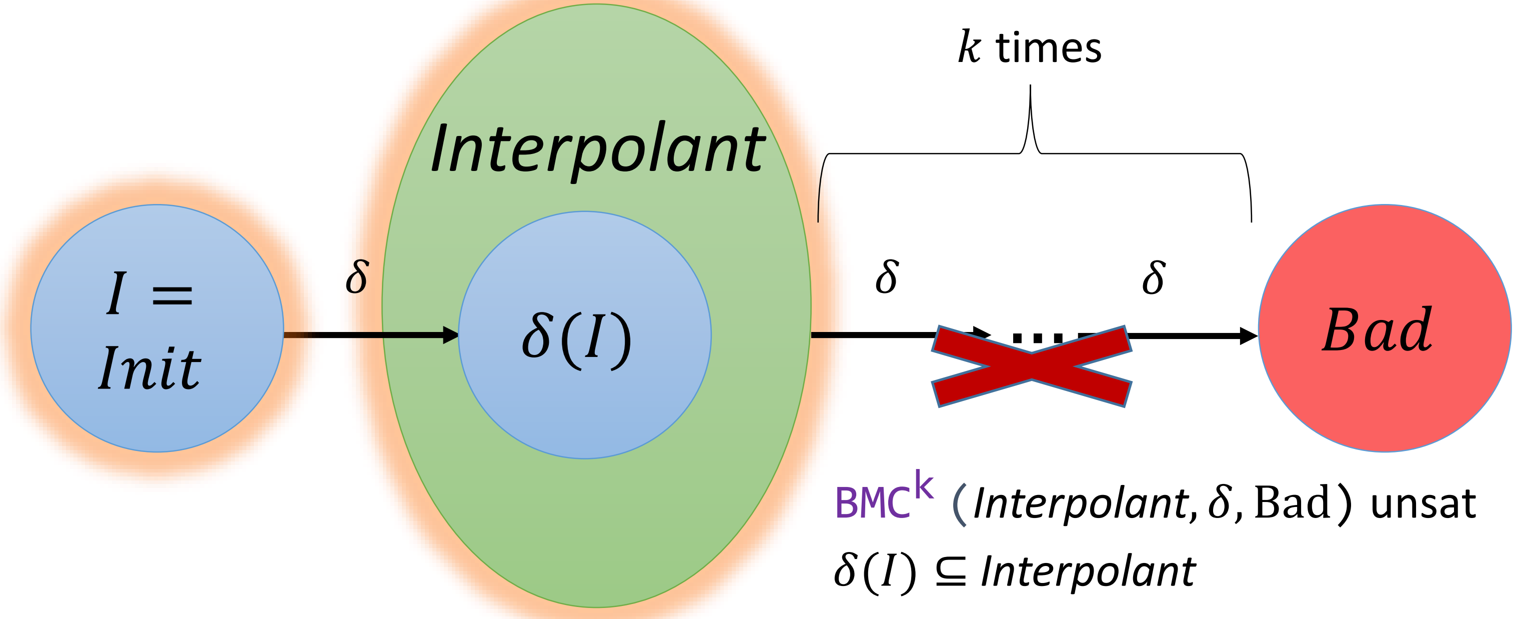
BMC^k $(Interpolant, \delta, \mathrm{Bad})$ unsat

$\delta(I) \subseteq Interpolant$

[CAV'03] Interpolation and SAT-Based Model Checking, McMillan

Interpolation-Based Inference

$$I = Init \vee Interpolant$$



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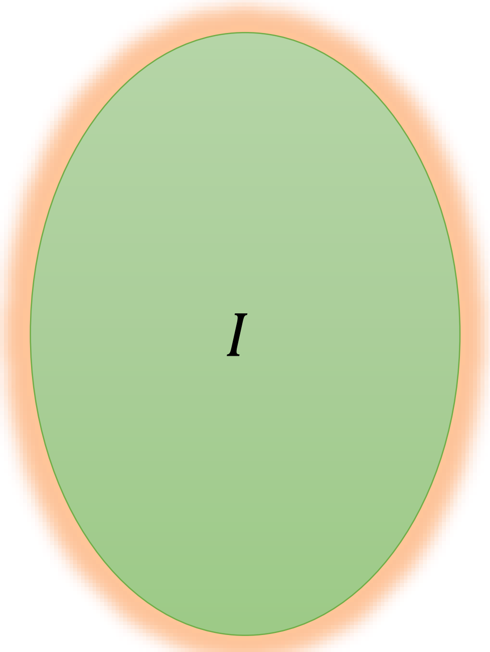
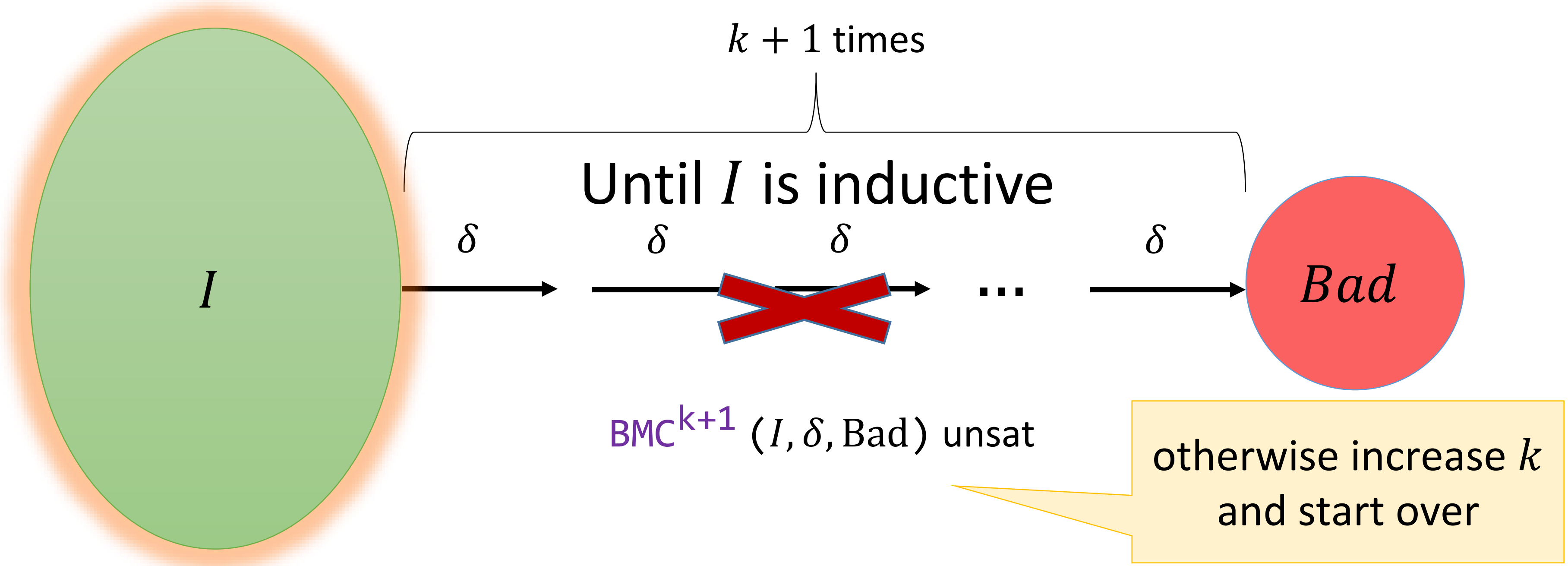


Inductive ?

[CAV'03] Interpolation and SAT-Based Model Checking, McMillan

Interpolation-Based Inference

$$I = Init \lor Interpolant \lor Interpolant_2 \lor \ \ldots$$



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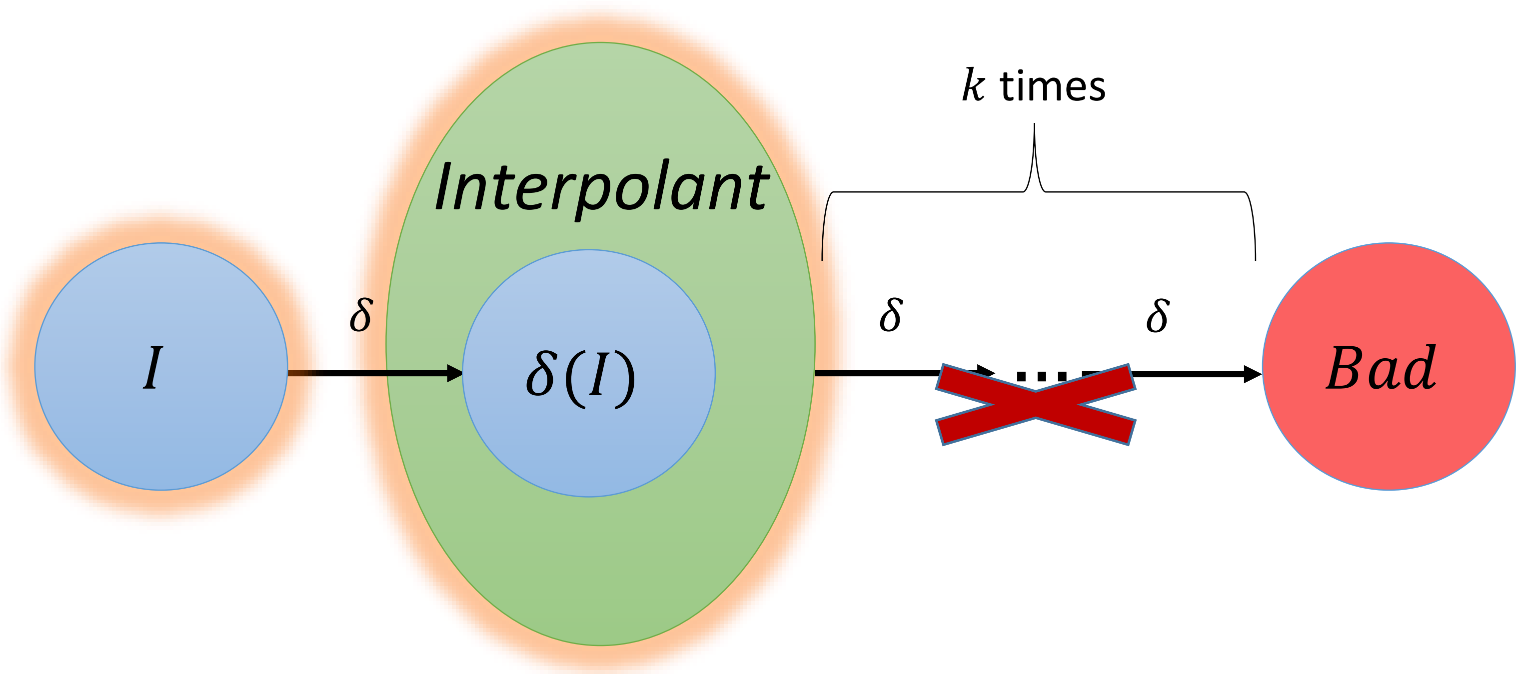
Computing an Interpolant



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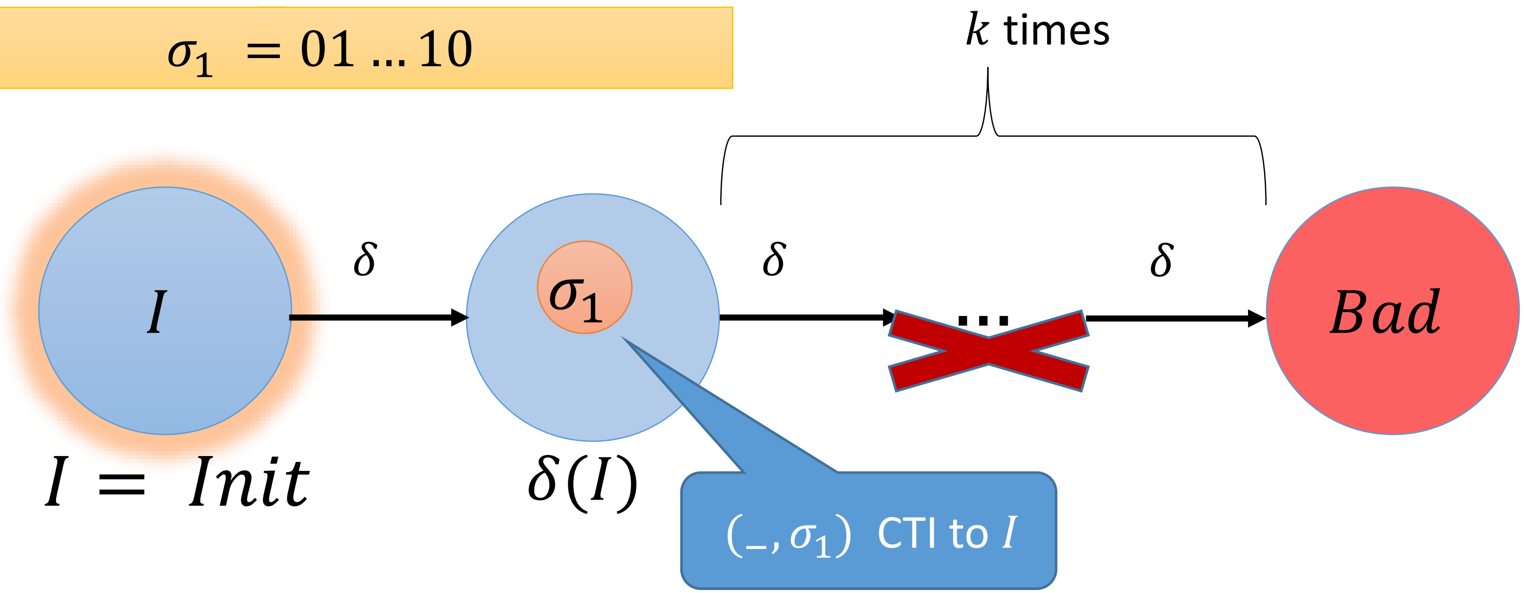
$$Interpolant_1 = (x_1 = 0 \wedge x_2 = 1 \wedge \cdots \wedge x_{n-1} = 1 \wedge x_n = 0)$$

$$\sigma_1 = 01 \ldots 10$$



[HVC'12] Computing Interpolants without Proofs. Chockler, Ivrii, Matsliah
[LPAR'13] Instantiations, Zippers and EPR Interpolation. Bjørner, Gurfinkel, Korovin, Lahav

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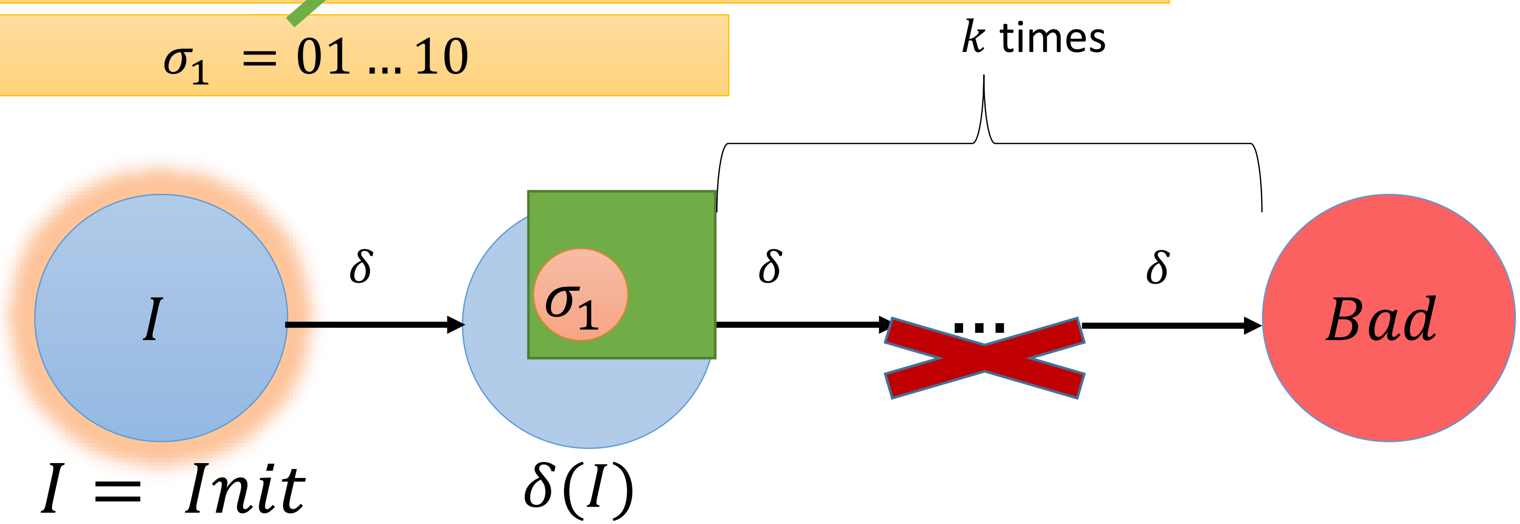

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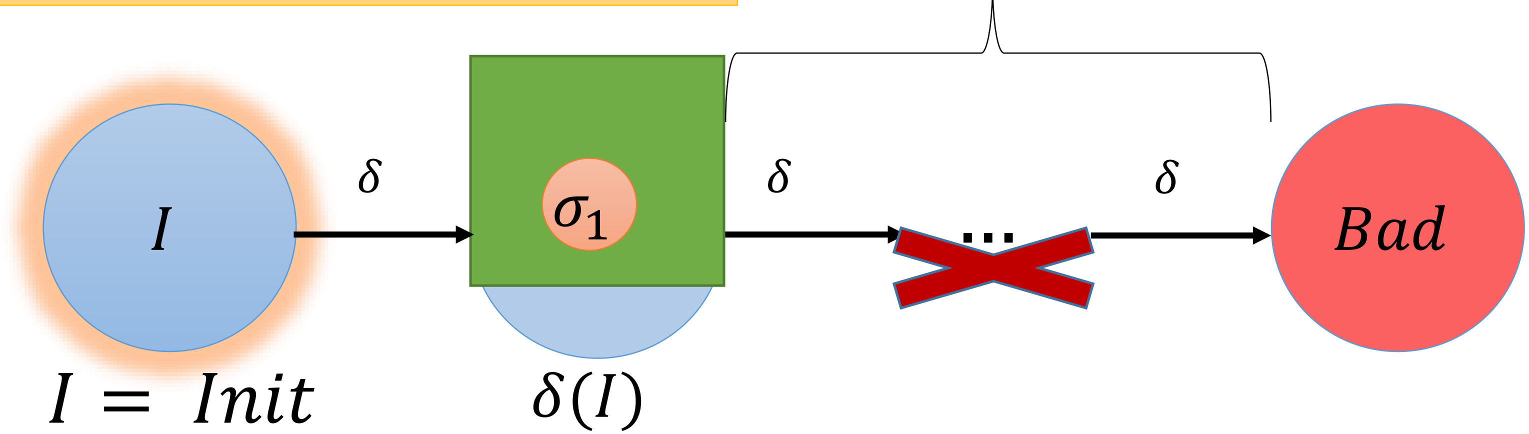

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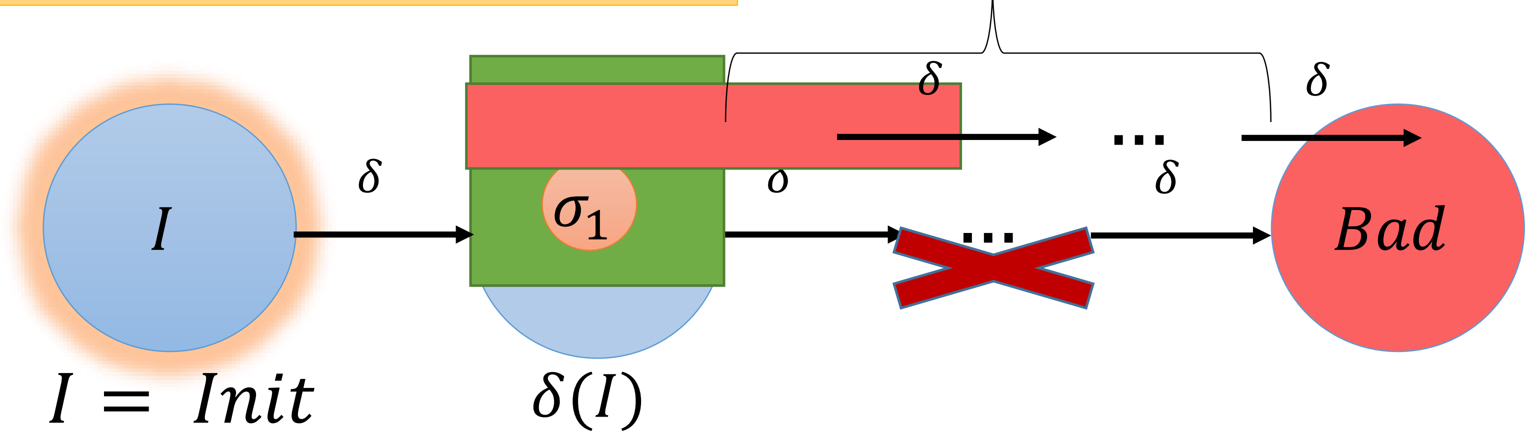
k times



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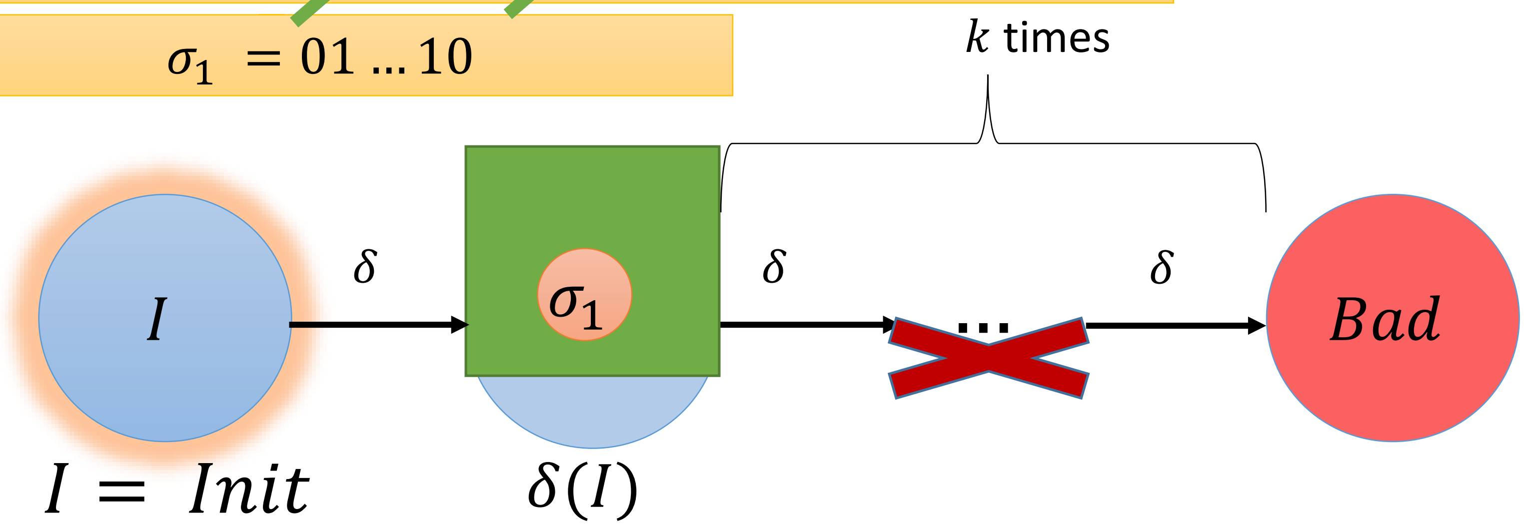

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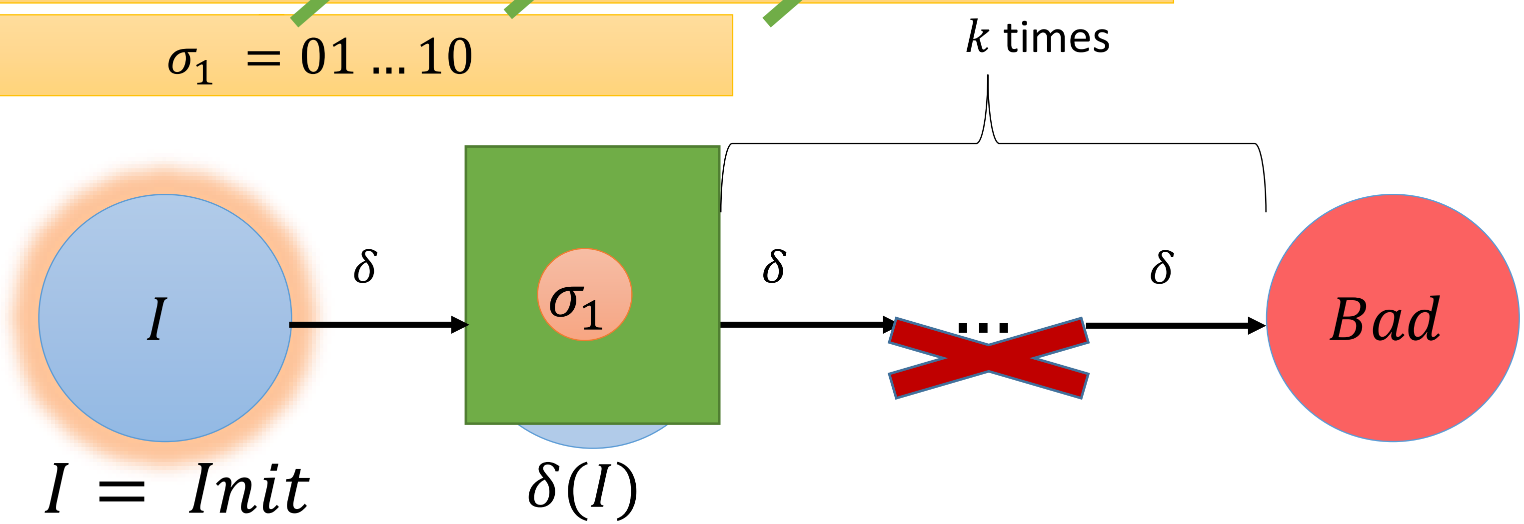

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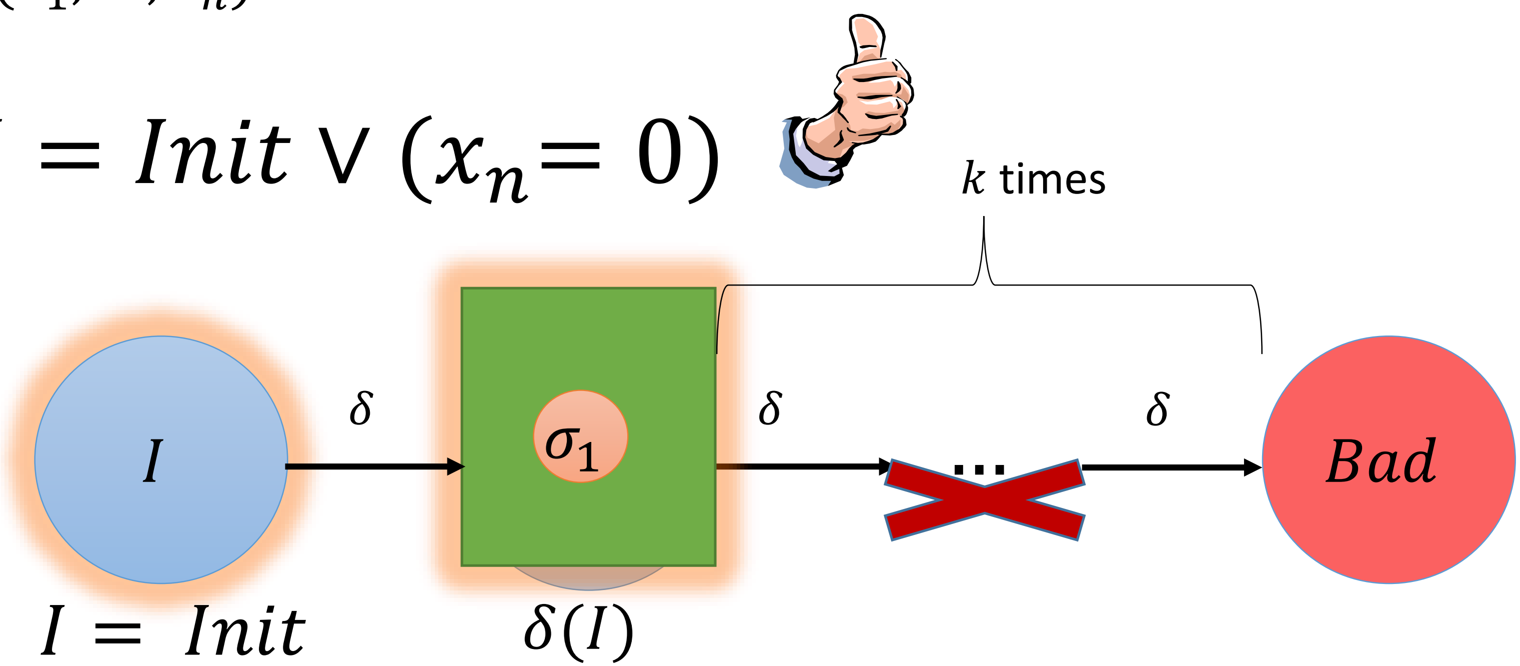
$$I = Init \vee (x_n = 0)$$



[HVC'12] Computing Interpolants without Proofs. Chockler, Ivrii, Matsliah

[LPAR'13] Instantiations, Zippers and EPR Interpolation. Bjørner, Gurfinkel, Korovin, Lahav

Model-Based Interpolation

Inferring invariant in DNF:

$$\underbrace{\left(\ell_1^1 \wedge \cdots \wedge \ell_{k_1}^1\right)}_{\text{gen}(\sigma_1)} \vee \ \ldots \vee \underbrace{\left(\ell_1^m \wedge \cdots \wedge \ell_{k_m}^m\right)}_{\text{gen}(\sigma_m)}$$

ITP-k:

I := false

`while` $(\_, \sigma')$ `counterexample`
`to` `Inductive`(δ, I):

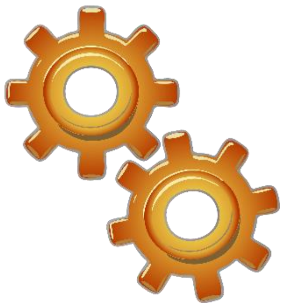
I := I ∨ **generalize**(σ')

generalize(σ'):

drop literals from σ'
while BMC$^k(\sigma', \delta, \text{Bad})$ unsat

Inductiveness-Query Model

inference algorithm



α_1 inductive?

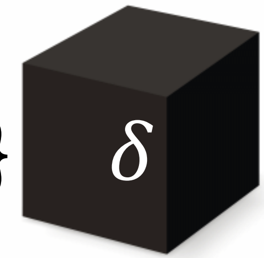
✓ / ✗+counterexample

…

α_m inductive?

✓ / ✗+counterexample

inductiveness-query oracle

$\{\alpha_i\}$  $\{\alpha_i\}$

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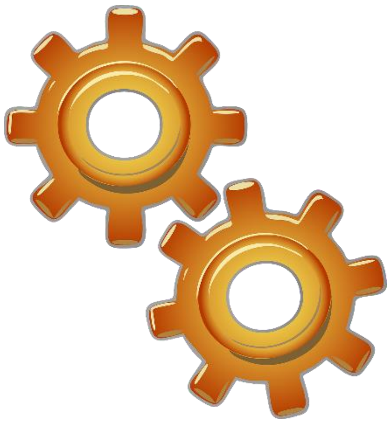
I inductive?

✓ / ✗

?

Hoare-Query Model

inference algorithm



Hoare-query oracle

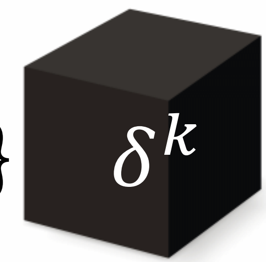
$\{\alpha_1\}\delta^k\{\beta_1\}?$

✓ / ✗ +counterexample

…

$\{\alpha_m\}\delta^k\{\beta_m\}?$

✓ / ✗ +counterexample

$\{\alpha_i\}$ 

$\{\beta_i\}$

$\mathrm{BMC}^k\ (\alpha_i, \delta, \neg\beta_i)$ unsat?

Trace $(\sigma_0, \ldots, \sigma_k)$ of δ s.t.
$\sigma_0 \vDash \alpha_i, \quad \sigma_k \vDash \neg\beta_i$

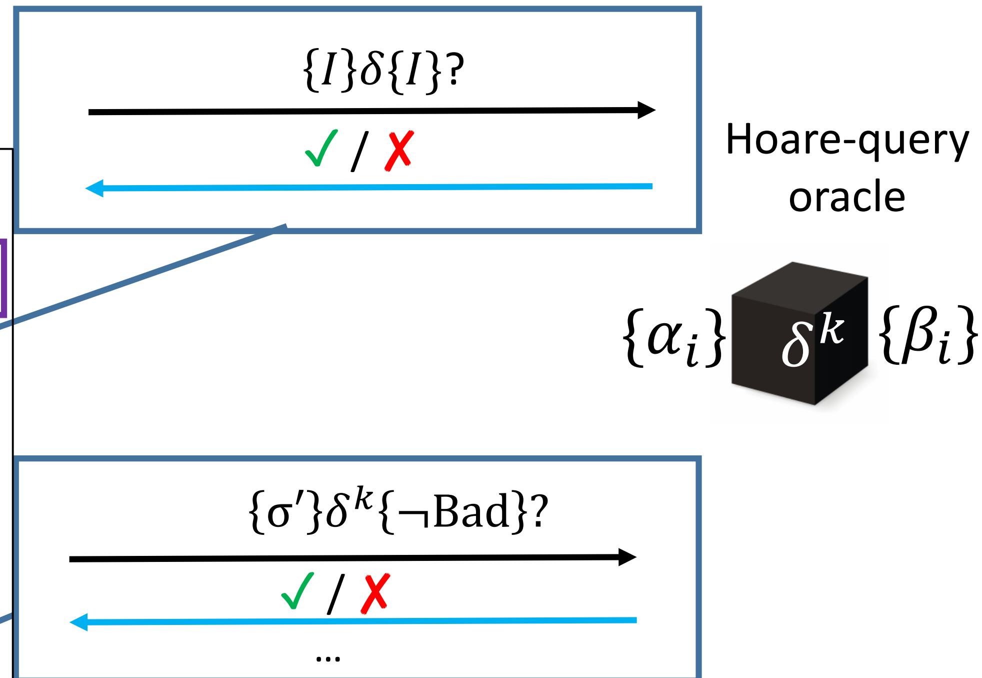
Capable of modeling several interesting algorithms

Hoare-Query Model

ITP-k:

I := false
while $(\_, \sigma')$ counterexample
    to Inductive(δ, I):
  I := $I \vee$ **generalize**(σ')

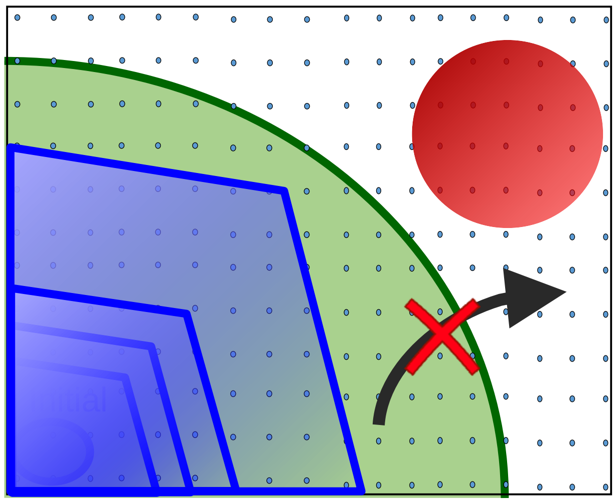
generalize(σ'):
  drop literals from σ'
  while BMC$^k(\sigma', \delta, \text{Bad})$ unsat



Also captures IC3/PDR

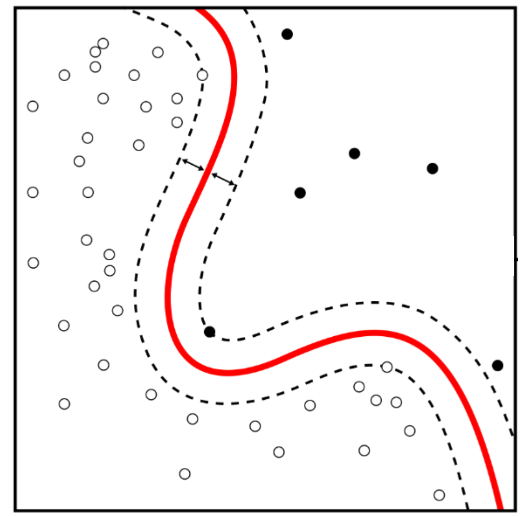
Outline

Invariant Inference



vs.

Exact Concept Learning



- Query-based learning models for invariant inference
- Complexity lower and upper bounds for each model
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from concept learning algorithms

Hoare-Query Complexity

Thm: Every Hoare-query algorithm requires $\mathbf{2^{\Omega(n)}}$ queries in the worst case for inferring $I \in$ DNF s.t. $|I| \leq \text{poly}(n)$

Throughout the talk

n is the vocabulary size, $k = \text{poly}(n)$

- even with unlimited computational power
- unconditional lower bound

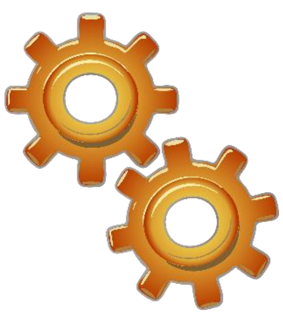
[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

Hoare-Query Complexity

<u>Thm</u>: Every Hoare-query algorithm requires $\mathbf{2^{\Omega(n)}}$ queries in the worst case for inferring $I \in$ DNF s.t. $|I| \leq \mathrm{poly}(n)$

<u>Proof sketch:</u>

given
inference algorithm
with $<2^{c \cdot n}$ queries



$\{\alpha_1\}\{\beta_1\}$?

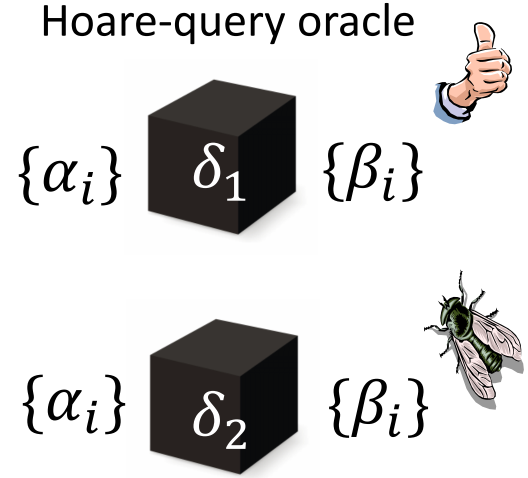
✓ / ✗

...

$\{\alpha_m\}\{\beta_m\}$?

✓ / ✗

construct δ_1, δ_2 for
Hoare-query oracle



1. δ_1 has an inductive invariants with at most n cubes
2. δ_2 does not (in fact, unsafe)
3. all queries return the same answer for δ_1, δ_2

Hoare-Query Complexity

Thm: Every Hoare-query algorithm requires $\mathbf{2^{\Omega(n)}}$ queries in the worst case for inferring $I \in$ DNF s.t. $|I| \leq \mathrm{poly}(n)$

Proof sketch:

given

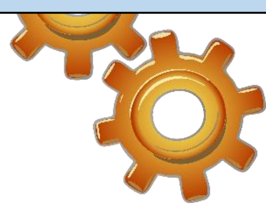
infe

wit

construct δ_1, δ_2 for

take δs checking validity of Boolean quantified formulas
$\exists x_1, \ldots, x_n.\ \forall y_1, \ldots, y_n.\ \phi(x_1, \ldots, x_n, y_1, \ldots, y_n)$

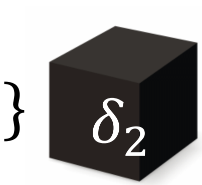
a sub-exponential number of valuations do not determine validity!



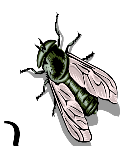
…

$\{\alpha_m\}\{\beta_m\}$?

✓ / ✗

$\{\alpha_i\}$ 

$\{\beta_i\}$



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Hoare-Query Complexity

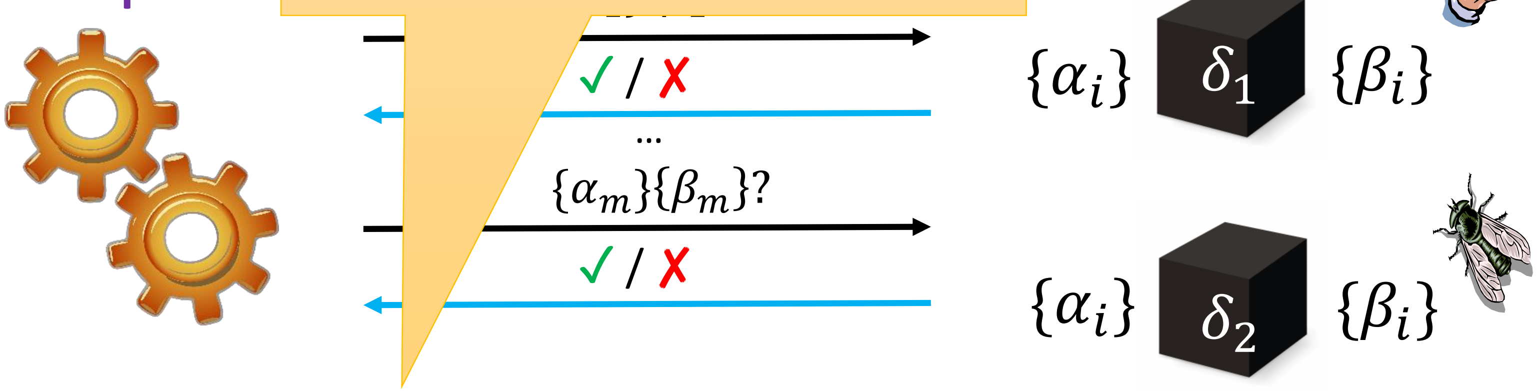
Thm: Every Hoare-query algorithm requires $\mathbf{2^{\Omega(n)}}$ queries in the worst case for inferring $I \in$ DNF s.t. $|I| \leq \mathrm{poly}(n)$

Proof sketch:

given inference algorithm **with <$2^{c \cdot n}$queries**

I is monotone: propositions appear only positively

construct δ_1, δ_2 for Hoare-query oracle



1. δ_1 has an inductive invariants with at most n cubes
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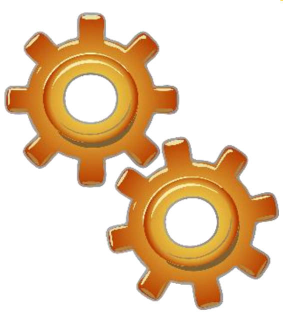
3. all queries return the same answer for δ_1, δ_2

Hoare-Query Complexity

Thm: Every Hoare-query algorithm requires $2^{\Omega(n)}$ queries in the worst case for inferring $I \in$ DNF s.t. $|I| \leq \mathrm{poly}(n)$

Proof sketch:

given inference algorithm **with <$2^{c \cdot n}$ queries**



I is monotone: propositions appear only positively

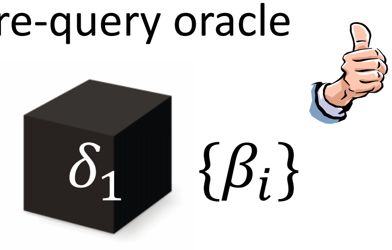
✓ / ✗

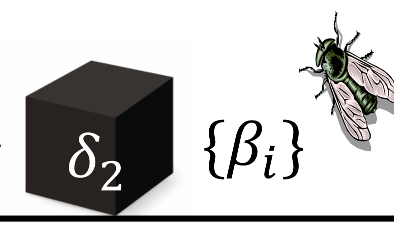
...

$\{\alpha_m\}\{\beta_m\}$?

✓ / ✗

construct δ_1, δ_2 for Hoare-query oracle

$\{\alpha_i\}$  $\{\beta_i\}$

$\{\alpha_i\}$  $\{\beta_i\}$

Cor: Every Hoare-query algorithm requires $2^{\Omega(n)}$ queries in the worst case for inferring short monotone DNF invariants

Hoare > Inductiveness

Thm: There exists a class of transition systems $\mathcal{P}$, so that for solving inference:

1. ∃ Hoare-query algorithm (with k=1) with $\mathrm{poly}(n)$ queries
2. ∀ inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

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1. ∃ Hoare-query algorithm (with k=1) with $\mathrm{poly}(n)$ queries
2. ∀ inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

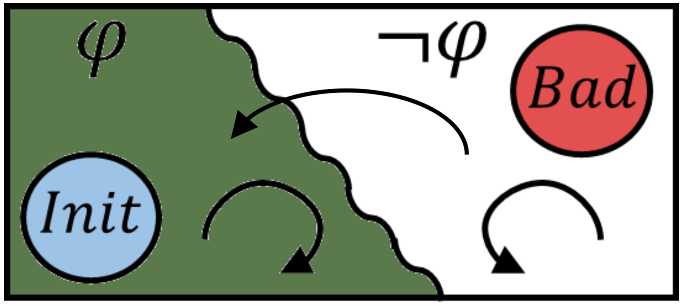
Proof:

$\mathcal{P}$ = maximal transition systems for monotone DNF with n cubes

propositions appear only positively

$$\varphi = x_1 \vee (x_2 \wedge x_3)$$

Maximal system for φ:



Hoare > Inductiveness

Upper bound:

∃ Hoare-query algorithm (with k=1) with $\mathrm{poly}(n)$ queries

Proof: **ITP-1** takes $\boldsymbol{O}(\boldsymbol{n}^{\mathbf{2}})$ queries

I := false

while $(\_,\sigma')$ counterexample to Inductive(δ, I):

  I := I ∨ **generalize**(σ')

generalize(σ'):

  drop literals from σ'

  while BMC[1]$(\sigma', \delta, \mathrm{Bad})$ unsat

minimal

$\sigma' \Rightarrow \varphi$

φ is monotone

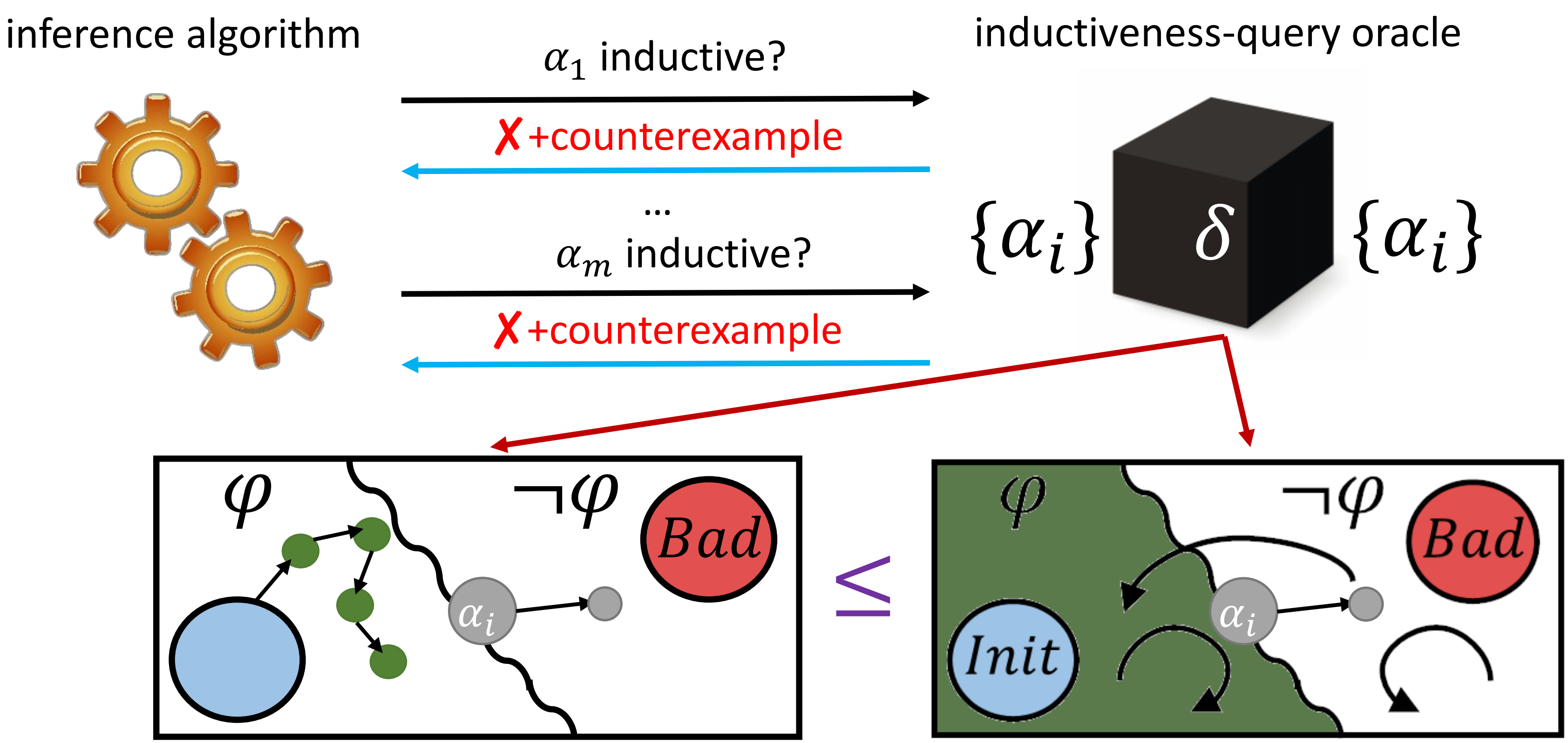
$$\varphi = \underbrace{x_1}_{\text{1 iteration}} \vee \underbrace{(x_2 \wedge x_3)}_{\text{1 iteration}}$$

Hoare > Inductiveness

<u>Lower bound</u>:

∀ inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

<u>Proof:</u>



Hoare > Inductiveness

Lower bound:

∀ inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

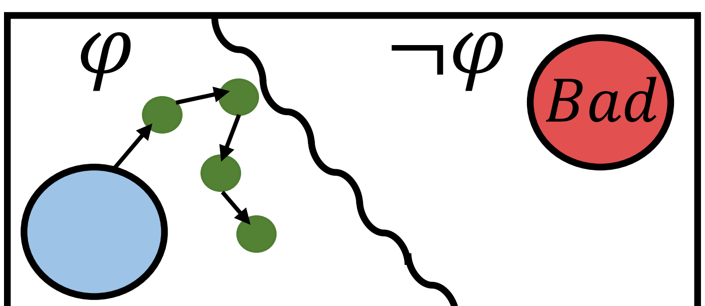
Proof:

previous Corollary

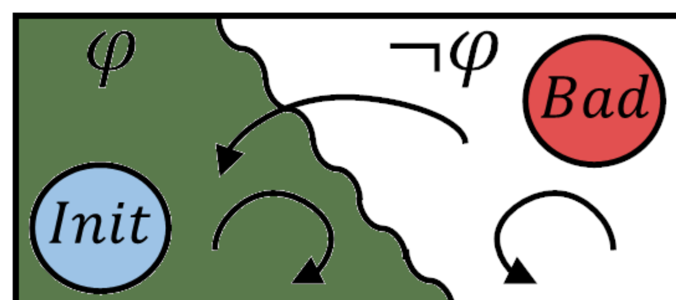
general systems
monotone DNF invariants

maximal systems
monotone DNF invariants

$2^{\Omega(n)} \leq$



$\leq$



Hoare > Inductiveness

<u>Thm</u>: There exists a class of transition systems $\mathcal{P}$, so that for solving inference:

1. ∃ Hoare-query algorithm (with k=1) with $\mathrm{poly}(n)$ queries
2. ∀ inductiveness-query algorithm requires $2^{\Omega(n)}$ queries

Similar proof works with a simple case of IC3/PDR

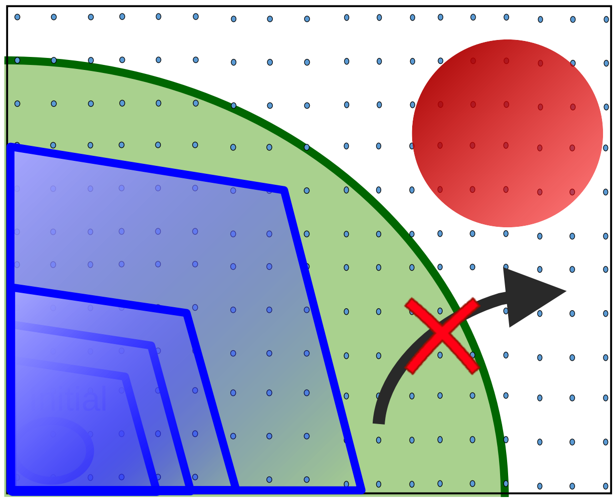
$\Rightarrow$ ICE cannot model PDR,

and the extension of [VMCAI'17] is necessary

[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv
[VMCAI'17] IC3 - Flipping the E in ICE. Vizel, Gurfinkel, Shoham, Malik.

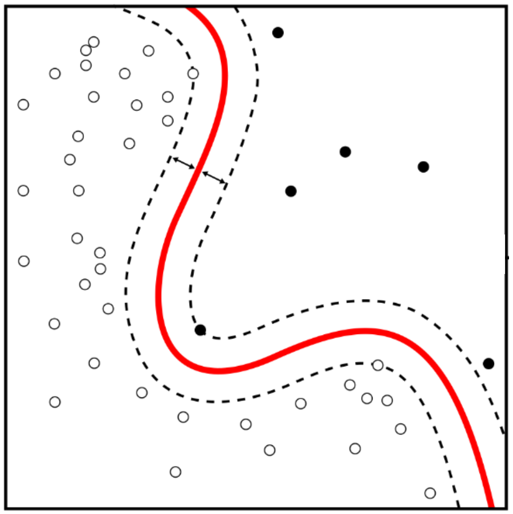
Outline

Invariant Inference



vs.

Exact Concept Learning



- Query-based learning models for invariant inference
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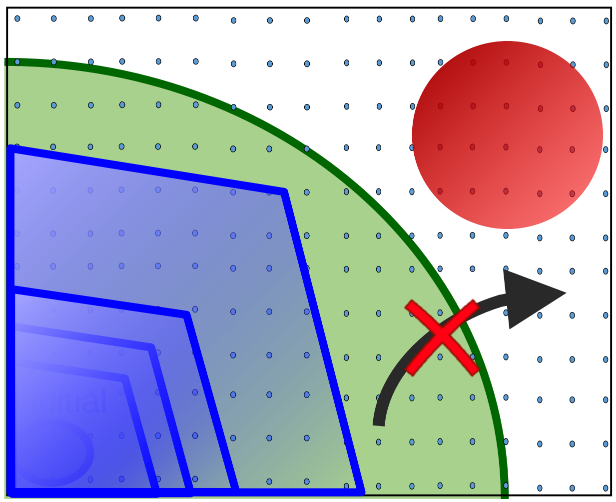
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from concept learning algorithms

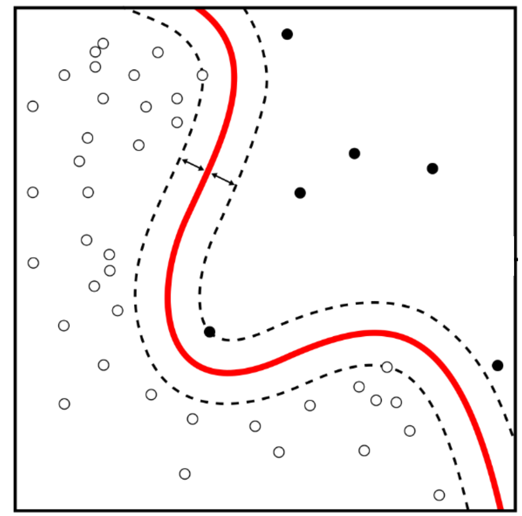
Inferring Monotone DNF

Invariant Inference



vs.

Exact Concept Learning



	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$2^{\Omega(n)}$

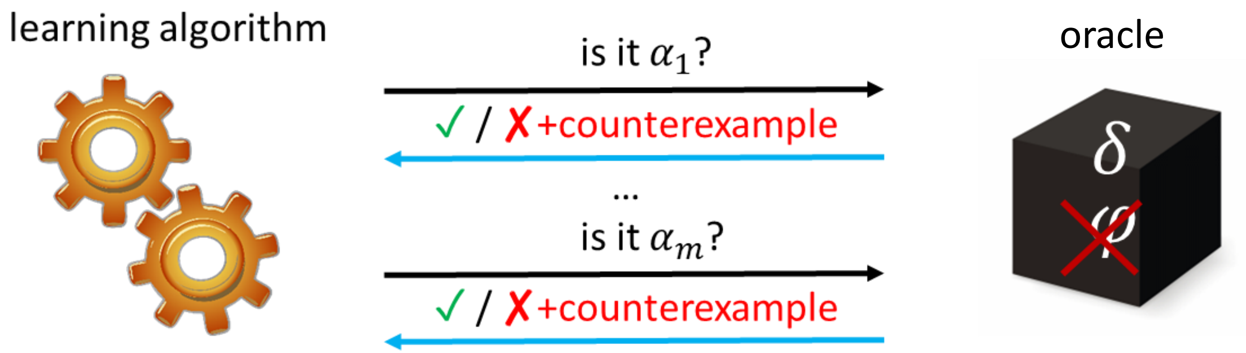
Equiv	sub-exponential
Equiv + mem	poly

[ML'87] Queries and Concept Learning, Angluin
[COLT'12] Tight Bounds on Proper Equivalence Query Learning of DNF, Hellerstein et al.

Inductiveness vs. Equivalence Queries

Invariant Inference

Exact Concept Learning



<u>Counterexamples to induction:</u>

$\sigma \vDash \neg\varphi$ **or** $\sigma' \vDash \varphi$

	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$2^{\Omega(n)}$

<u>Positive/negative examples:</u>

$\sigma^+ \vDash \varphi$, $\sigma^- \vDash \neg\varphi$

Equiv	sub-exponential
Equiv + mem	poly

[ML'87] Queries and Concept Learning, Angluin

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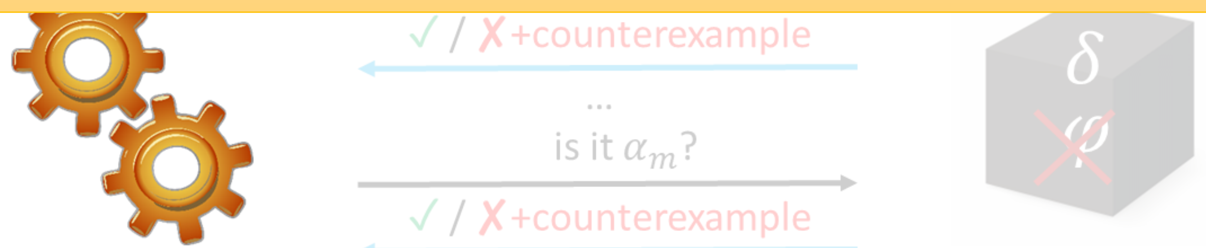
Inductiveness vs. Equivalence Queries

<u>Thm</u>: Learning from counterexamples to induction is **harder** than learning from positive/negative examples.



<u>Counterexamples to induction:</u>

$\sigma \models \neg\varphi$ **or** $\sigma' \models \varphi$

	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$2^{\Omega(n)}$

<u>Positive/negative examples:</u>

$\sigma^+ \models \varphi\ ,\ \sigma^- \models \neg\varphi$

Equiv	sub-exponential
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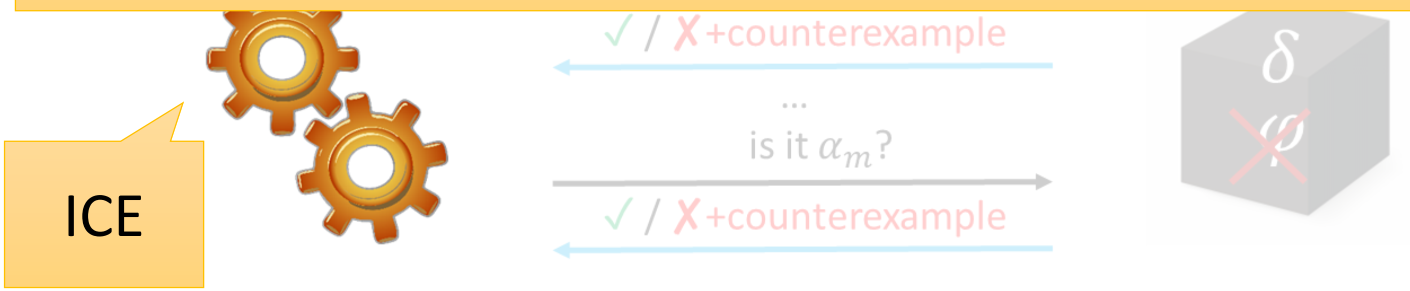

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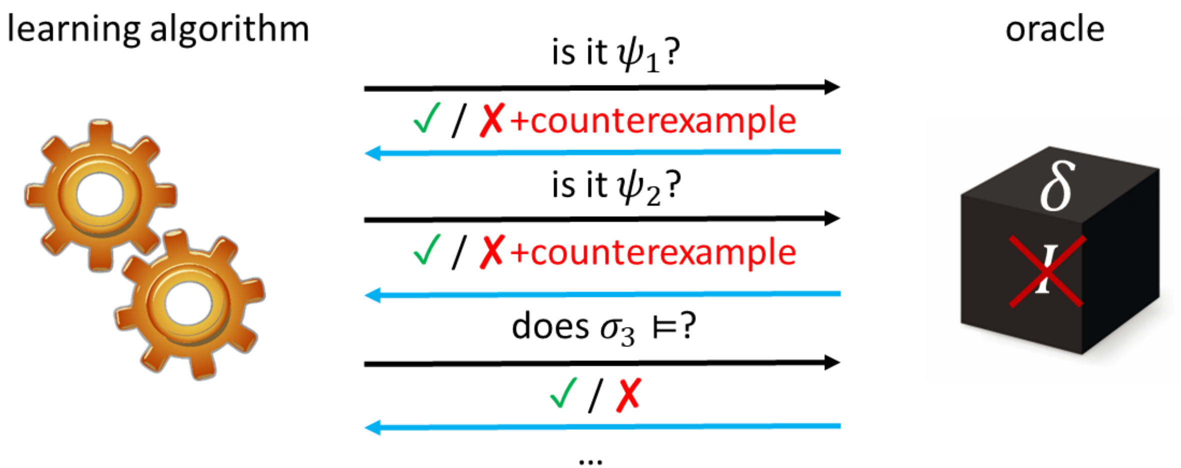
[ML'87] Queries and Concept Learning, Angluin
[COLT'12] Tight Bounds on Proper Equivalence Query Learning of DNF, Hellerstein et al.
[CAV'14] ICE: A Robust Framework for Learning Invariants. Garg, Löding, Madhusudan, Neider

Invariant Inference with Equivalence & Membership Queries



Invariant Inference

	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$\boxed{2^{\Omega(n)}}$

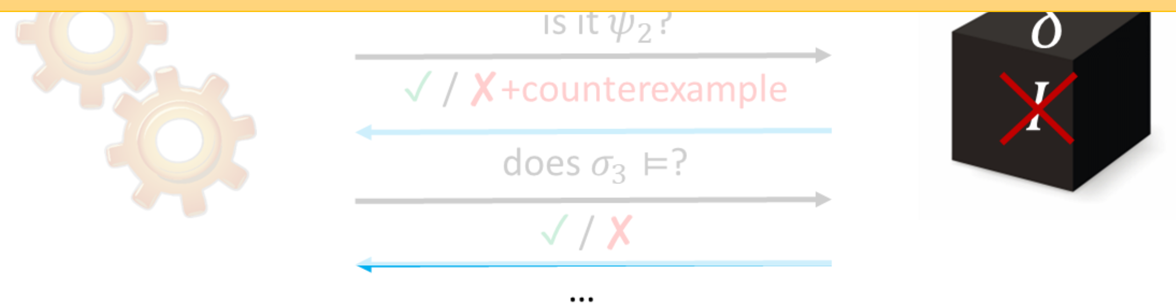
Exact Concept Learning

Equiv	sub-exponential
Equiv + mem	$\boxed{\text{poly}}$

[ML'87] Queries and Concept Learning, Angluin
[COLT'12] Tight Bounds on Proper Equivalence Query Learning of DNF, Hellerstein et al.

Invariant Inference with Equivalence & Membership Queries

<u>Thm</u>. In general, in the Hoare-query model, **no efficient way** to implement a teacher for equivalence and membership queries



Invariant Inference

	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$\boxed{2^{\Omega(n)}}$

Exact Concept Learning

Equiv	sub-exponential
Equiv + mem	$\boxed{\text{poly}}$

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[POPL'20] Complexity and Information in Invariant Inference. Feldman, Immerman, Shoham, Sagiv

Invariant Inference with Equivalence & Membership Queries

Thm. In general, in the Hoare-query model, **no efficient way** to implement a teacher for equivalence and membership queries

Sufficient conditions for

exact learning algorithms $\Longrightarrow$ invariant inference algorithms

	Maximal	General
Inductive	$2^{\Omega(n)}$	$2^{\Omega(n)}$
Hoare	poly	$\boxed{2^{\Omega(n)}}$

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[ML'87] Queries and Concept Learning, Angluin
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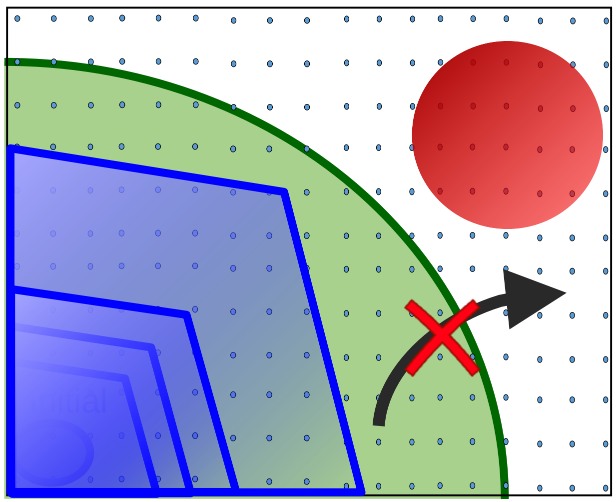
Outline

Invariant Inference



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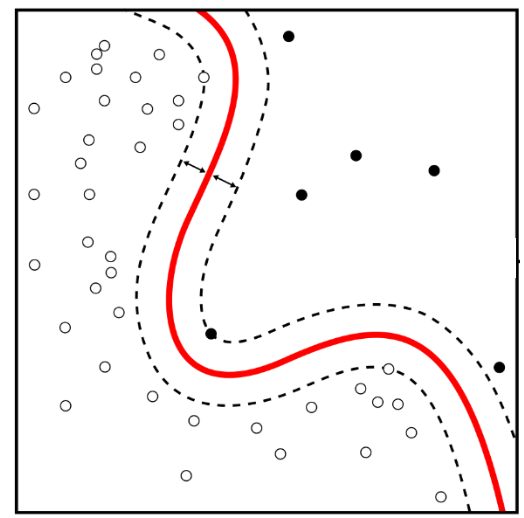
Exact Concept Learning



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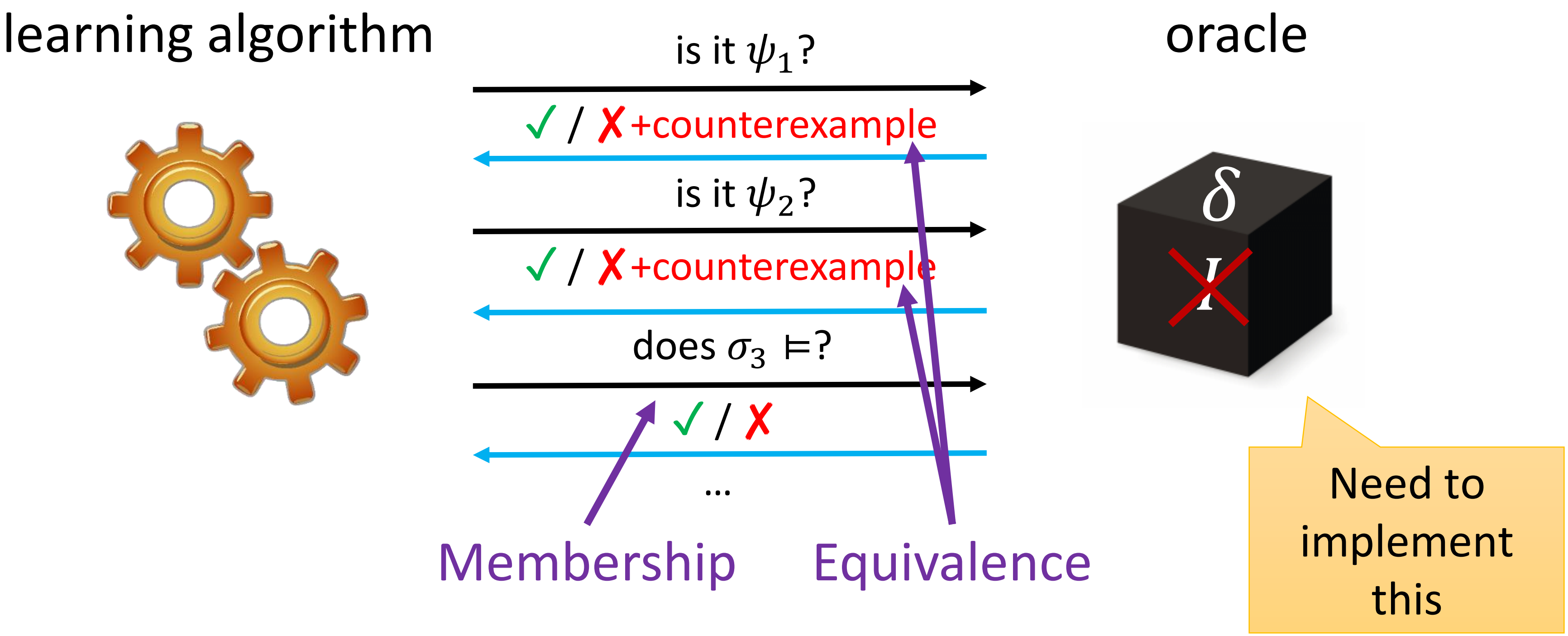
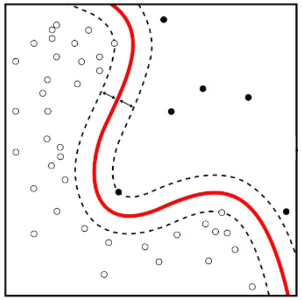
From Learning to Inference



From Learning to Inference



Exact **learning**
DNF formulas

ψ := false

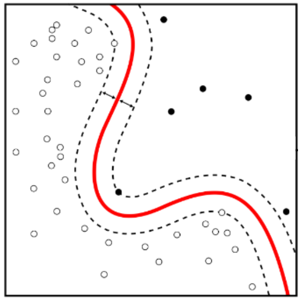
while σ' counterexample
    to Equivalence(ψ):

  ψ := $\psi \lor \mathbf{generalize}(\sigma')$

$\mathbf{generalize}(\sigma')$:
  drop literals from σ'
  while Membership(σ')= ✓

[CACM'84] A Theory of the Learnable. Valiant
[ML'87] Queries and Concept Learning. Angluin
[ML'95] On the Learnability of Disjunctive Normal Form Formulas. Aizenstein and Pitt

From Learning to Inference



Exact **learning** DNF formulas $\Longrightarrow$

ψ := false

while σ' counterexample
to **Equivalence**(ψ): → **Inductive**(I)

ψ := $\psi \lor$ **generalize**(σ')

generalize(σ'):
drop literals from σ'
while **Membership**(σ')= ✓ → $\mathrm{BMC}^k(\sigma', \delta, \mathbf{Bad})$ **unsat**

[CACM’84] A Theory of the Learnable. Valiant
[ML’87] Queries and Concept Learning. Angluin
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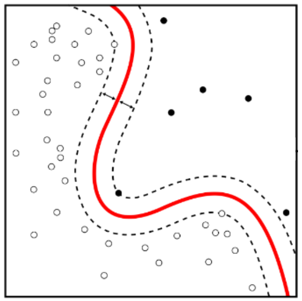
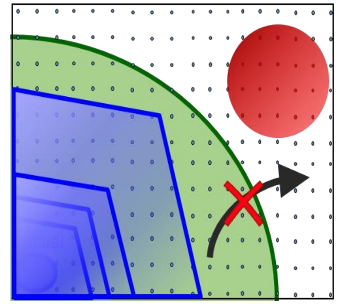
From Learning to Inference



Exact **learning**
DNF formulas

$\Longrightarrow$



Inferring
DNF invariants

ψ := false

while σ' counterexample
 to **Equivalence**(ψ):
  ψ := $\psi \vee$ **generalize**(σ')

generalize(σ'):
  drop literals from σ'
  while **Membership**(σ')= ✓

I := false

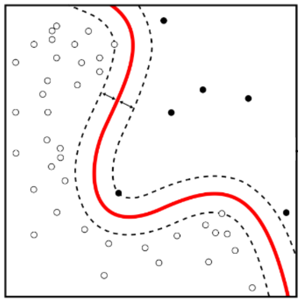
while $(\_,\sigma')$ counterexample
 to **Inductive**(I):
  I := $I \vee$ **generalize**(σ')

generalize(σ'):
  drop literals from σ'
  while $\mathbf{BMC}^k(\sigma', \delta, \mathbf{Bad})$ **unsat**

[CACM’84] A Theory of the Learnable. Valiant
[ML’87] Queries and Concept Learning. Angluin
[ML’95] On the Learnability of Disjunctive Normal Form Formulas. Aizenstein and Pitt

[CAV’03] Interpolation and SAT-Based Model Checking, McMillan
[HVC’12] Computing Interpolants without Proofs. Chockler, Ivrii, Matsliah

From Learning to Inference

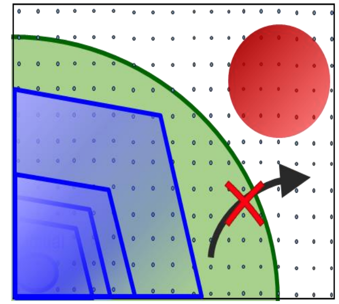


Efficiently

Exact **learning**

DNF formulas

$\Longrightarrow$



Efficiently

Inferring

DNF invariants

When is the transformation correct?

ψ := false

while σ' counte... to **Equiva**...

ψ := $\psi \vee$ **generalize**(σ')

generalize(σ'):

drop literals from σ'

while **Membership**(σ')= ✓

I := false

...') counterexample to **Inductive**(I):

I := $I \vee$ **generalize**(σ')

generalize(σ'):

drop literals from σ'

while **BMC**^{k}(σ', δ, **Bad**) **unsat**

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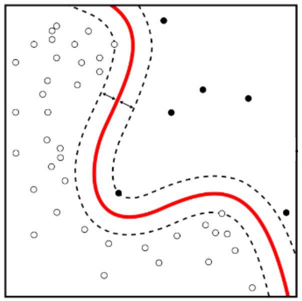
From Learning to Inference

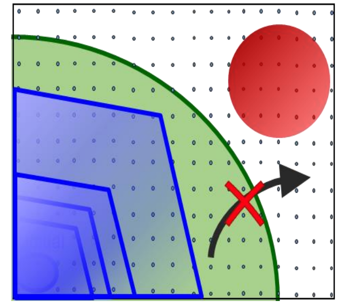


Efficiently

Exact **learning**

DNF formulas

$\Longrightarrow$



Efficiently

Inferring

DNF invariants

Thm: can implement queries when
the invariant is k-fenced
and the algorithm's queries are one-sided

generalize(σ'):
  drop literals from σ'
  while **Membership**$(\boldsymbol{\sigma'})$= ✓

$\longrightarrow$

generalize(σ'):
  drop literals from σ'
  while $\mathbf{BMC}^{\mathbf{k}}(\boldsymbol{\sigma'}, \boldsymbol{\delta}, \mathbf{Bad})$ **unsat**

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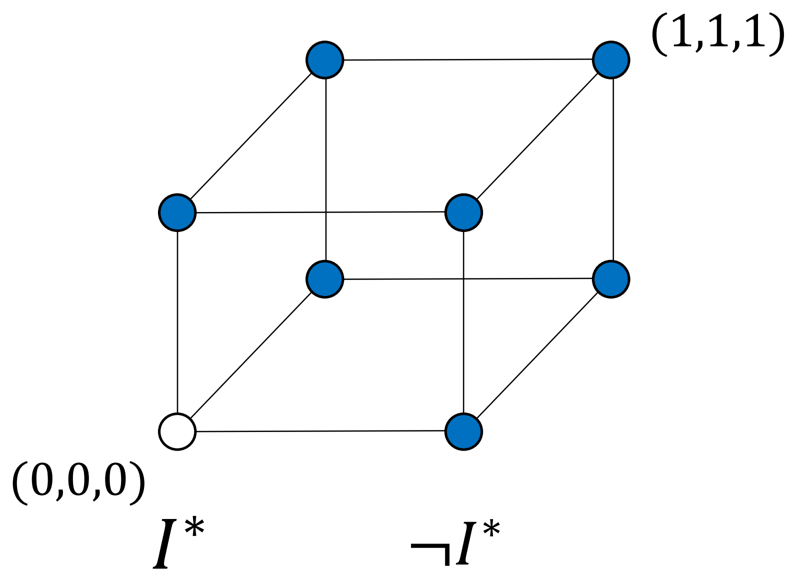
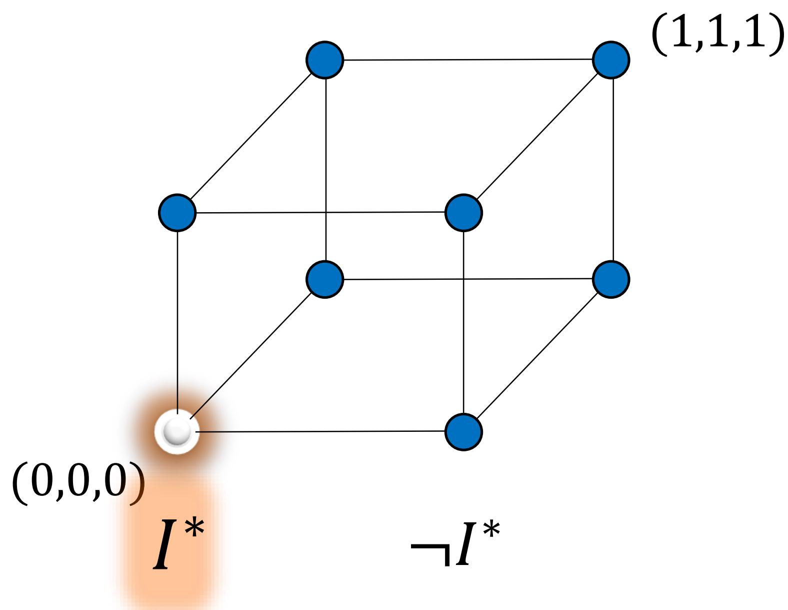
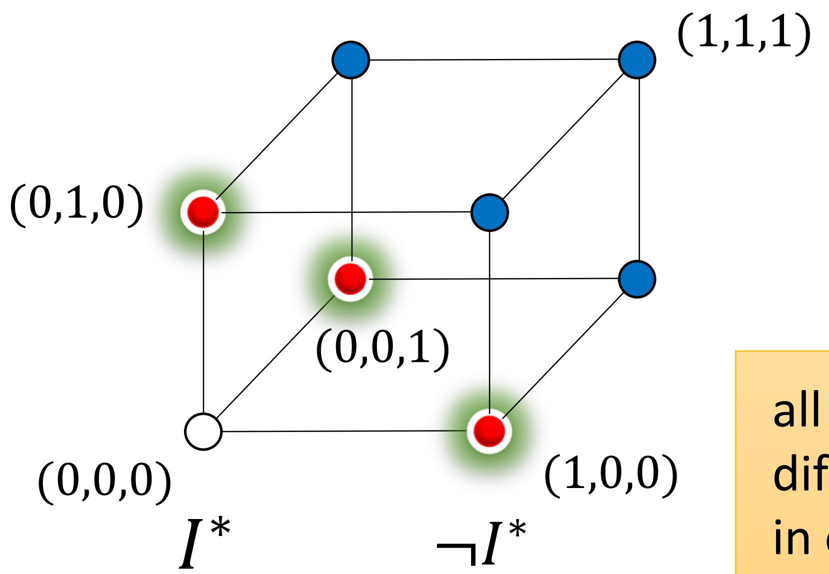
k-Fenced Invariants



k-Fenced Invariants



k-Fenced Invariants

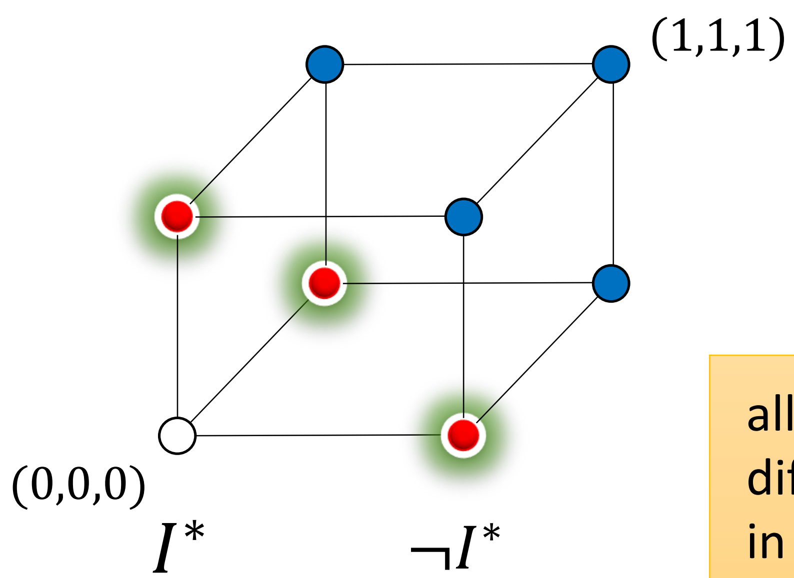


all states $\sigma \notin I^*$ that differ from some $\sigma' \in I^*$ in one bit

$$\partial^-(I^*)$$

Outer boundary

k-Fenced Invariants



all states $\sigma \notin I^*$ that differ from some $\sigma' \in I^*$ in one bit

I^* is k-fenced if

all the states in $\partial^-(I^*)$

can reach a bad state in at most k steps

Example: k-Fenced Invariant

Init:
$(x_1, \dots, x_n) \coloneqq 0 \dots 0$

Bad:
$(x_1, \dots, x_n) = 1 \dots 1$

δ:
$y_1, \dots, y_n \coloneqq *$
$x_1, \dots, x_n \coloneqq (x_1, \dots, x_n) +$
$2 \cdot (y_1, \dots, y_n) \pmod{2^n}$

$$I^*: \; x_n \neq 1$$

all the states in $\partial^-(I^*) = \{x_n = 1\}$
can reach a bad state in at most k steps $= 1$

Example: k-Fenced Invariant

Init:
$(x_1, \dots, x_n) \;\coloneqq\; 0 \dots 0$

Bad:
$(x_1, \dots, x_n) \;=\; 1 \dots 1$

δ:
$y_1, \dots, y_n \;\coloneqq\; *$
$x_1, \dots, x_n \;\coloneqq\; (x_1, \dots, x_n) + 2 \cdot (y_1, \dots, y_n) \pmod{2^n}$

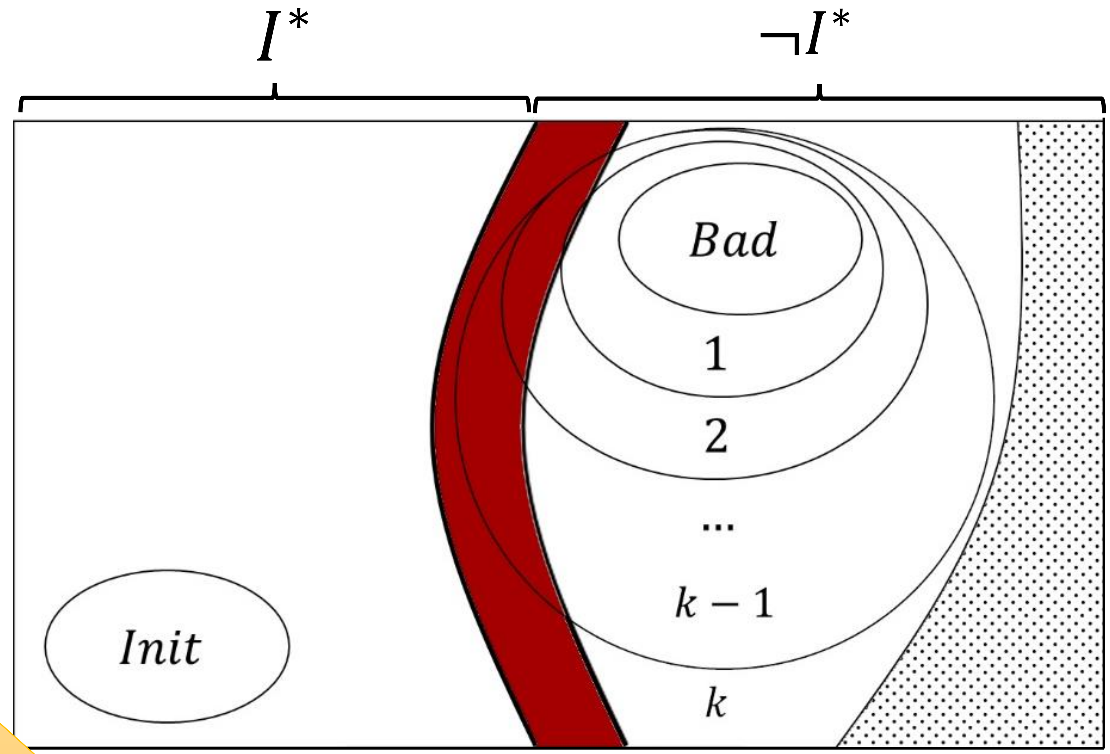
In general **not** all states in $\neg I^*$ need to reach bad

$I^*\!: \;\; x_n \neq 1$

In this example $\neg I^*$

all the states in $\partial^-(I^*) \;=\; \{x_n = 1\}$
can reach a bad state in at most k steps $= 1$

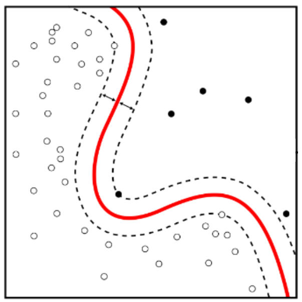
k-Fenced Invariants



In general **not** all states in $\neg I^*$ need to reach bad

all the states in $\partial^-(I^*)$
can reach a bad state in at most k steps

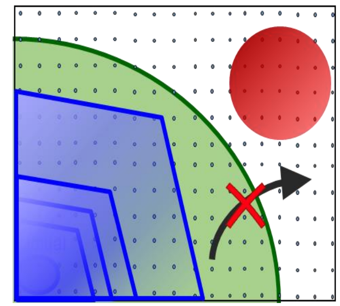
From Learning to Inference



Efficiently

Exact **learning**
DNF formulas

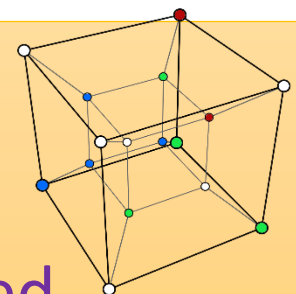
$\Longrightarrow$



Efficiently

Inferring
DNF invariants

Thm: can implement queries when
the invariant is k-fenced ✓
and the algorithm's queries are <u>one-sided</u>



`One-Sided Equivalence`(ψ): $\psi \Longrightarrow \varphi$

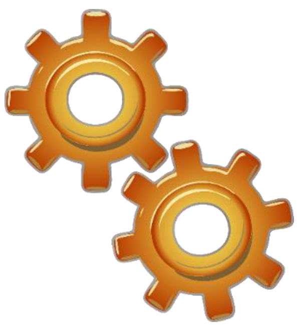
`One-Sided Membership`(σ): $\sigma \in \varphi \cup \partial^{-}(\varphi)$

One-Sided Equivalence Queries to Invariants

inference algorithm

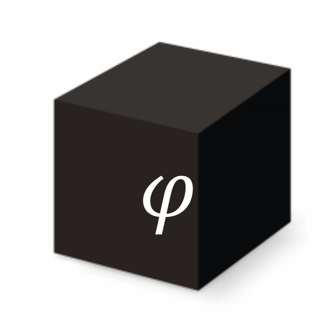
$\psi \Longrightarrow \varphi$

teacher



is it ψ?

✓ / ✗ +counterexample



is ψ an inductive invariant?

✓ yes hooray!

✗ +counterexample transition:

(σ, σ') s.t. $\sigma \vDash \psi, \sigma' \vDash \neg\psi$

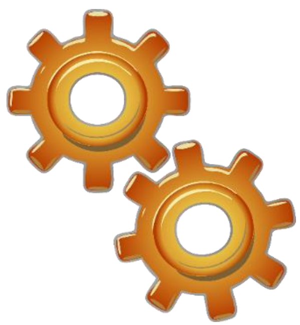
Always return σ' as positive example

One-Sided Membership Queries to k-Fenced Invariants

inference algorithm

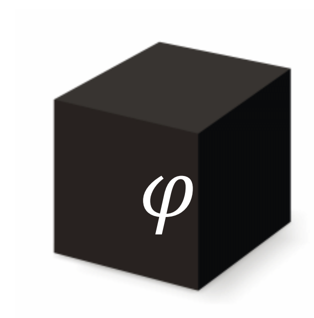
$\boldsymbol{\sigma \in \varphi \cup \partial^{-}(\varphi)}$

teacher



is $\sigma_3 \vDash$?

✓ / ✗



can't σ_3 reach bad states **in k steps?**

BMC^{k}$(\sigma_3, \delta, \mathrm{Bad})$ **unsat?**

✓ then yes

✗ then no

Doesn't always imply that $\sigma_3 \vDash I^*$

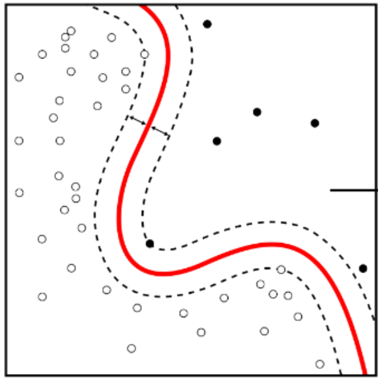
From Learning to Inference

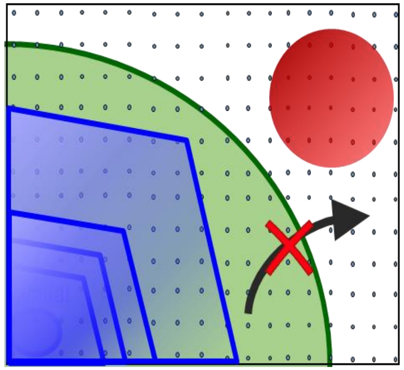
<u>Thm</u>: Let $\mathcal{C}$ be a class of formulas.

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many one-sided queries

$\Longrightarrow$

$\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced





[POPL'21] Learning the Boundary of Inductive Invariants. Feldman, Sagiv, Shoham, Wilcox

Efficient Inference

<u>Thm 1</u>: $\mathcal{C}$ = monotone DNF

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many one-sided queries $\implies$ $\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced

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Efficient Inference

Thm 1: $\mathcal{C}$ = monotone DNF

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many one-sided queries $\Longrightarrow$ $\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced

$\psi \Longrightarrow \varphi$

```
ψ := false
while σ′ counterexample
      to Equivalence(ψ):
   ψ := ψ ∨ generalize(σ′)
```

$\sigma' \in \varphi \cup \partial^-(\varphi)$

generalize(σ'):
drop literals from σ'
while **Membership**(σ')= ✓

```
I := false
while (_,σ′) counterexample
      to Inductive(I):
   I := I ∨ generalize(σ′)
```

generalize(σ'):
drop literals from σ'
while **BMC^k**(σ', δ, Bad) unsat

Efficient Inference

Thm 1: $\mathcal{C}$ = monotone DNF

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many one-sided queries $\implies$ $\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced

Thm 1: The **interpolation-based algorithm** converges in a polynomial number of SAT queries if I^* is

- k-fenced, and
- has a short monotone DNF representation

[CACM'84] A Theory of the Learnable. Valiant

[ML'87] Queries and Concept Learning. Angluin

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Efficient Inference

Thm 2: $\mathcal{C}$ = almost-monotone DNF

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many one-sided queries $\Longrightarrow$ $\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced

Thm 2: A **different algorithm** converges in a polynomial number of SAT queries if If I^* is

- k-fenced, and
- has a short almost-monotone DNF representation

at most $O(1)$ terms include negated variables

[Inf. Comput. '95] Exact Learning Boolean Function via the Monotone Theory. Bshouty

Inference from Unrestricted Queries

Thm': Let $\mathcal{C}$ be a class of formulas.

two-sided

$\exists\mathcal{A}$ **identifying** $\varphi \in \mathcal{C}$ with polynomially-many ~~one-sided~~ queries $\Longrightarrow$ $\exists\mathcal{A}$ **inferring** $I^* \in \mathcal{C}$ with polynomially-many SAT queries whenever I^* is k-fenced

Thm 3: A **different algorithm** converges in a polynomial number of SAT queries if I^* is

- two-sided k-fenced, and
- has a short DNF *and* a short CNF representation

e.g., I^* is expressible as a short decision tree

[Inf. Comput. '95] Exact Learning Boolean Function via the Monotone Theory. Bshouty

Inference from Unrestricted Queries

Thm’: Let $\mathcal{C}$ be a class of formulas.

two-sided

$\exists \mathcal{A}$ **inferring** $^* \in \mathcal{C}$ with
ia y-many
que ies
* is k-fenced

Thm: also when I^* is one-sided k-fenced
but **not by transformation from learning**

Thm 3: A **different algorithm** converges in a polynomial number of SAT queries if I^* is

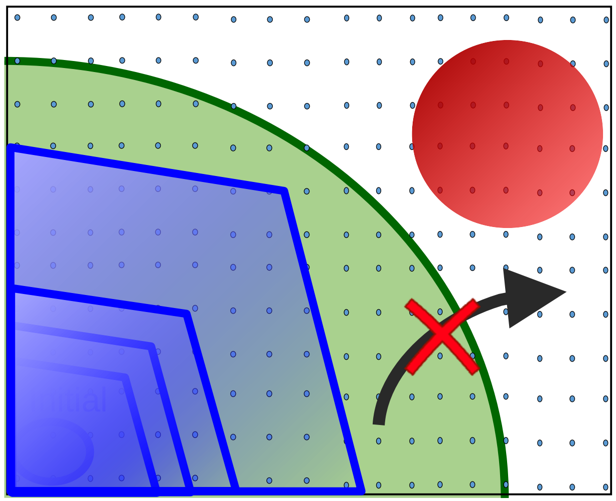
- two-sided k-fenced, and
- has a short DNF *and* a short CNF representation

e.g., I^* is expressible as a short decision tree

[Inf. Comput. ‘95] Exact Learning Boolean Function via the Monotone Theory. Bshouty
[SAS ‘22] Invariant Inference With Provable Complexity From the Monotone Theory. Feldman, Shoham

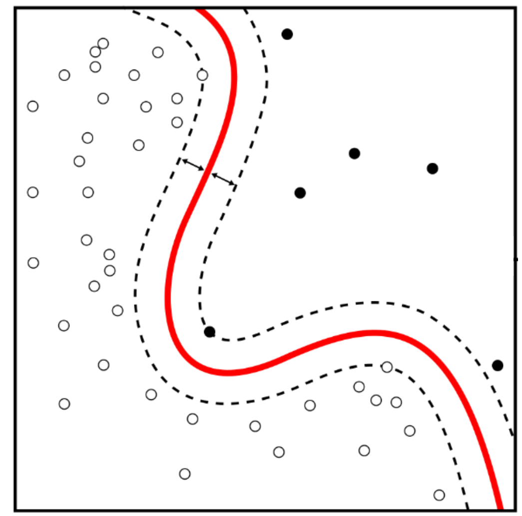
Conclusion (1)

Invariant Inference



vs.

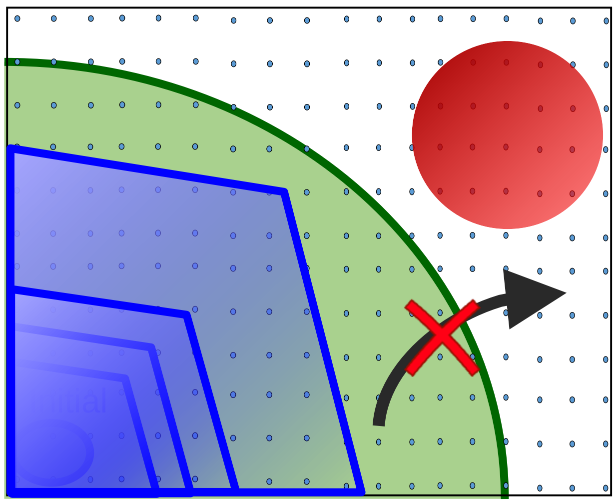
Exact Concept Learning



- Query-based learning models for invariant inference
- Complexity lower and upper bounds for each model
- Invariant inference is harder than concept learning
- Complexity results for invariant inference algorithms from concept learning algorithms

Conclusion (2)

Invariant Inference



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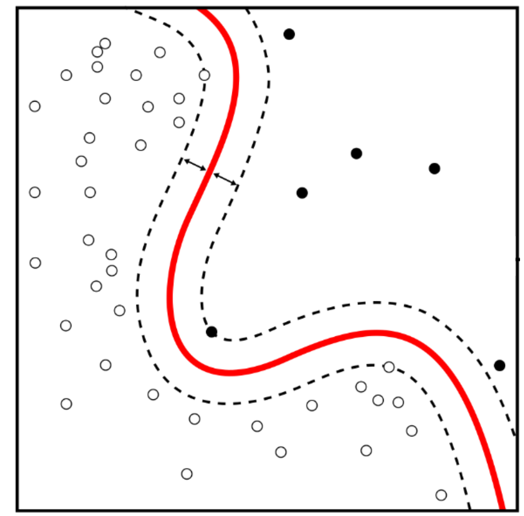

- What about IC3/PDR?
- Impact of k in the Hoare query model?
- Is the fence condition necessary?
- Other conditions?
- Beyond Boolean programs