Counterexample-Guided Repair in Boolean Functional Synthesis

Supratik Chakraborty

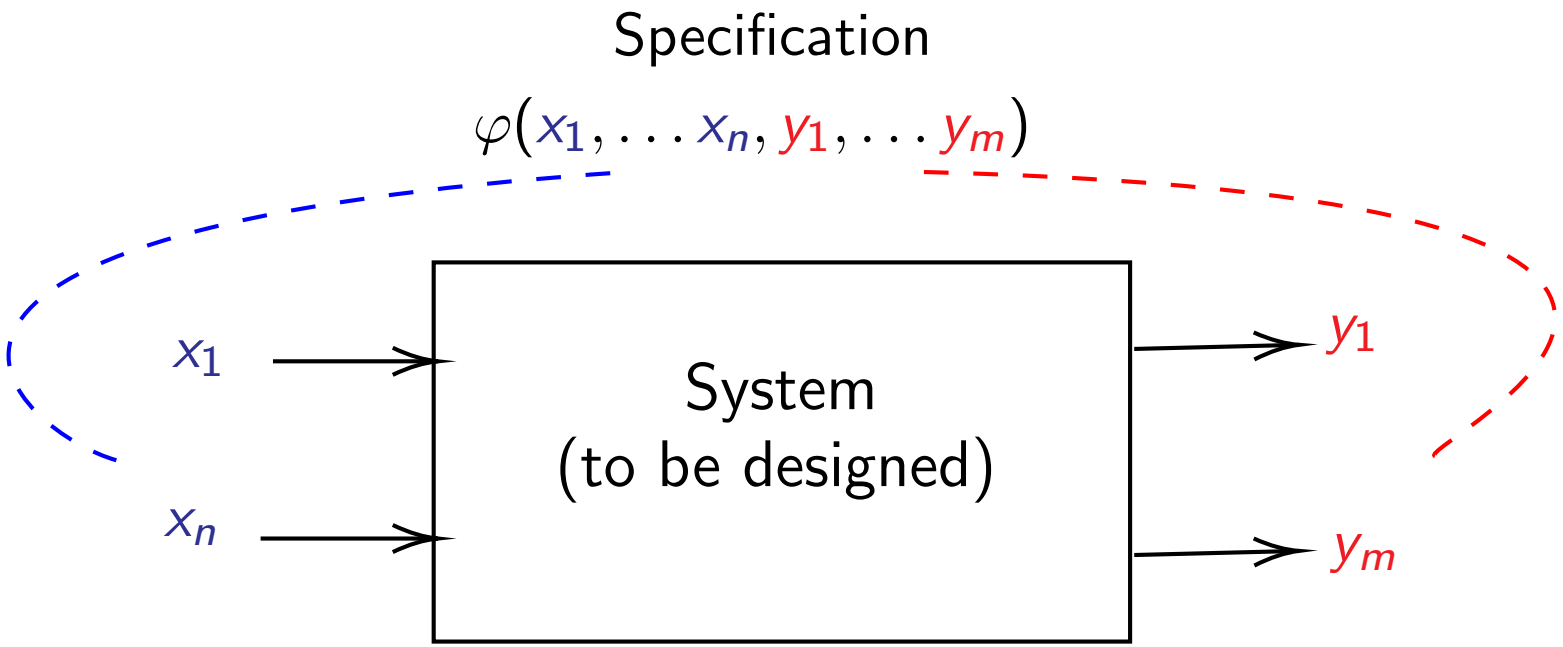
Indian Institute of Technology Bombay

Joint work with

S. Akshay, Shubham Goel, Ajith John, Sumit Kulal, Shetal Shah

Formal Methods Update Meeting, 2021

Synthesis: A Generic View



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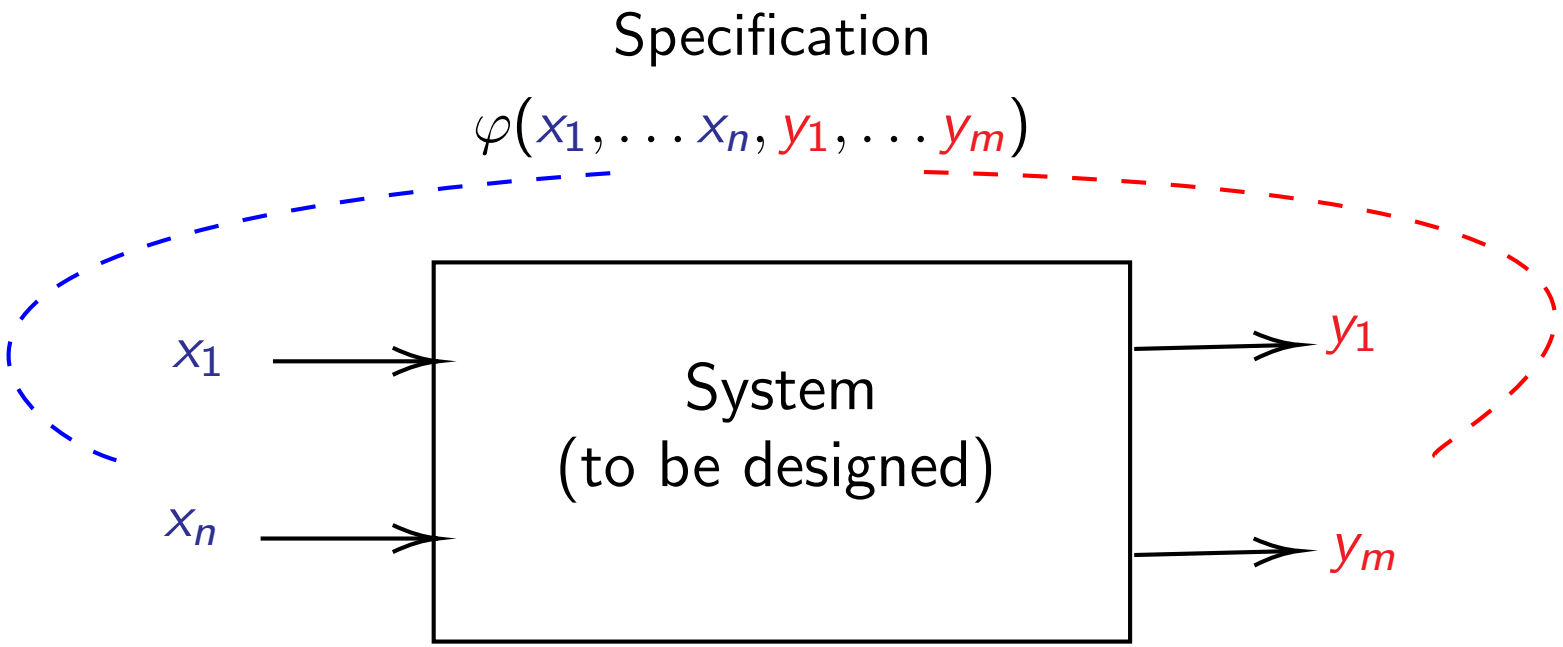

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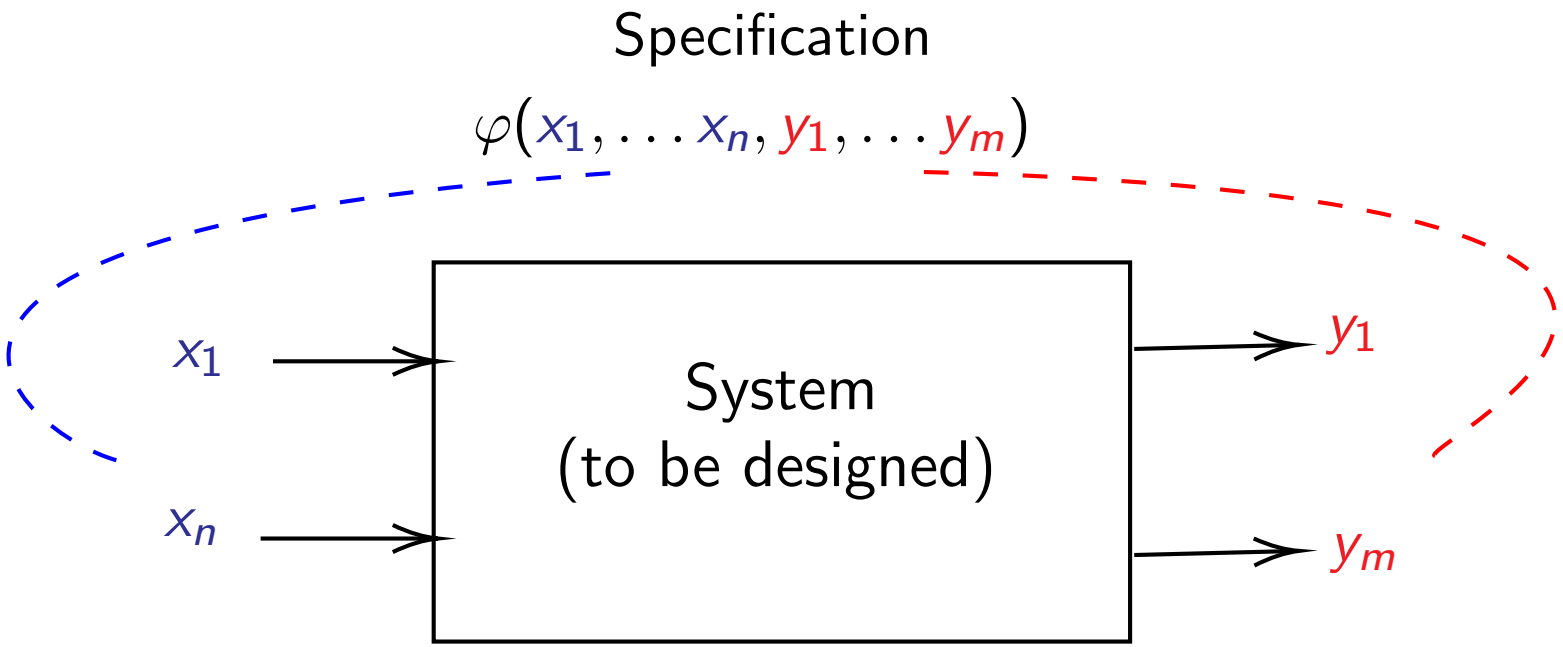

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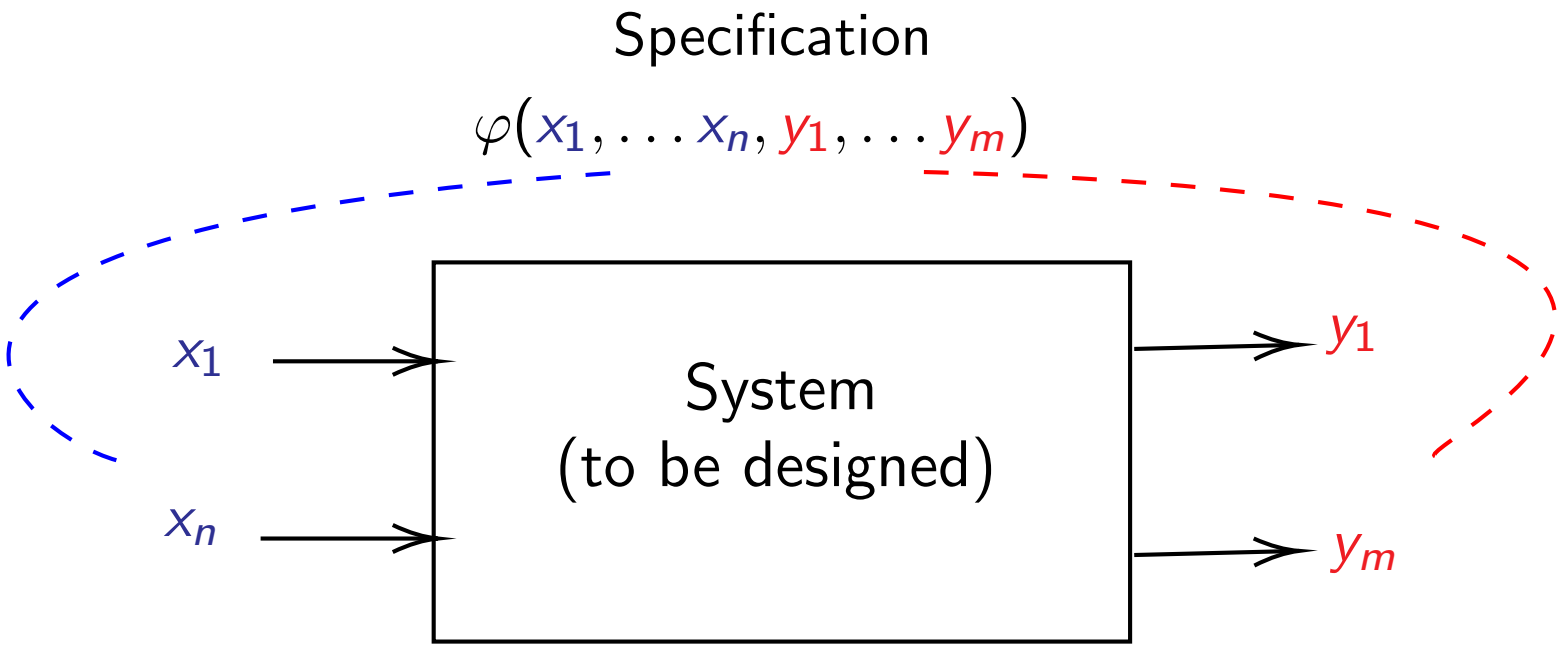

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  - x_i *input* variables (vector X)
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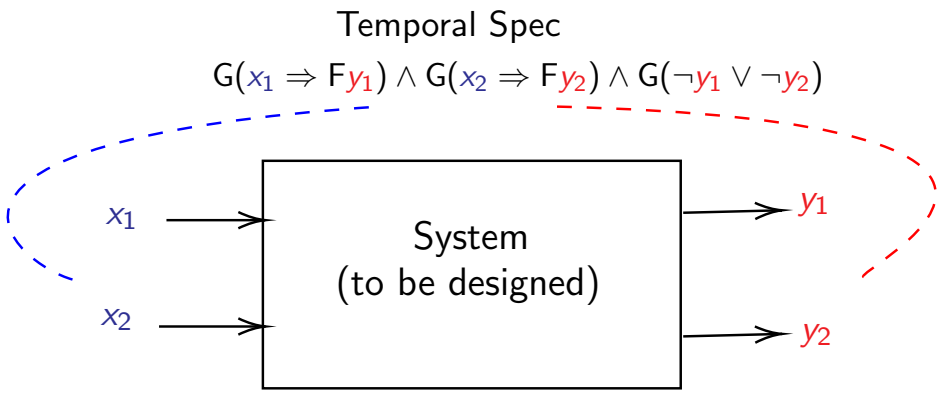
- Need Y as functions F of
  - "History" of X and Y, "State" of system, ...

  such that $\varphi(X, F)$ is satisfied.

Synthesis: Some Examples



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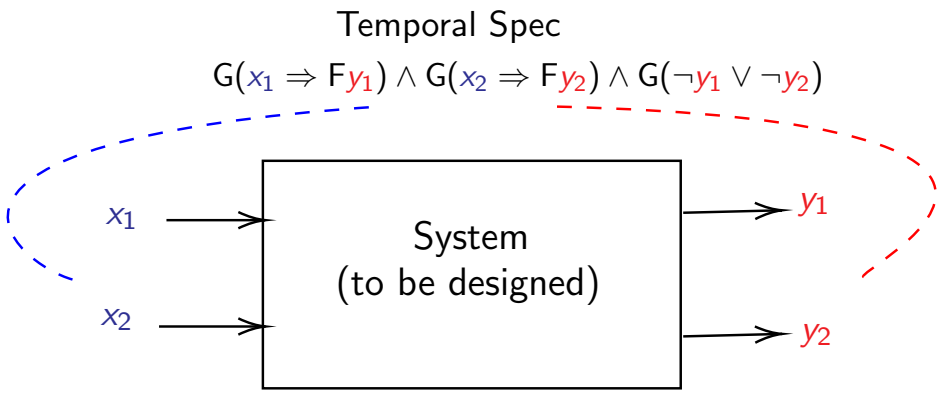

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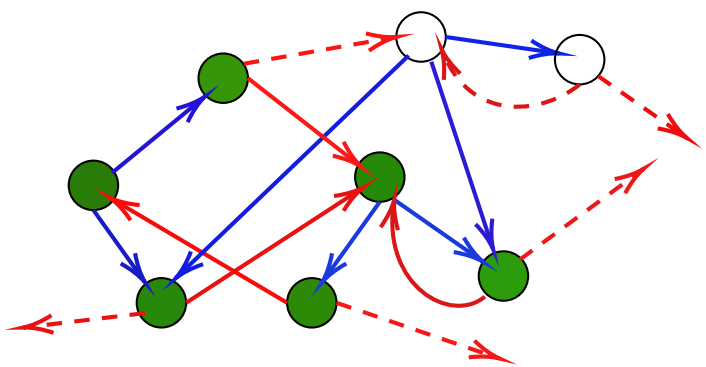

Winning Region

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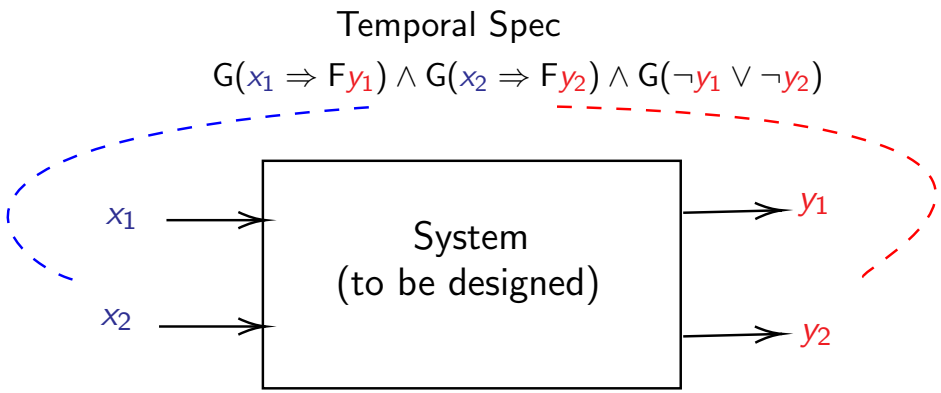

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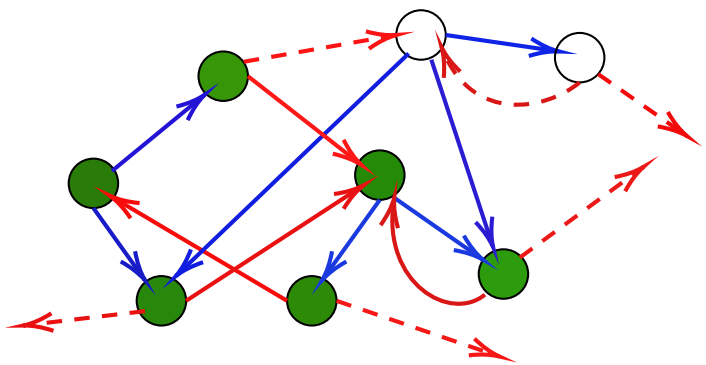

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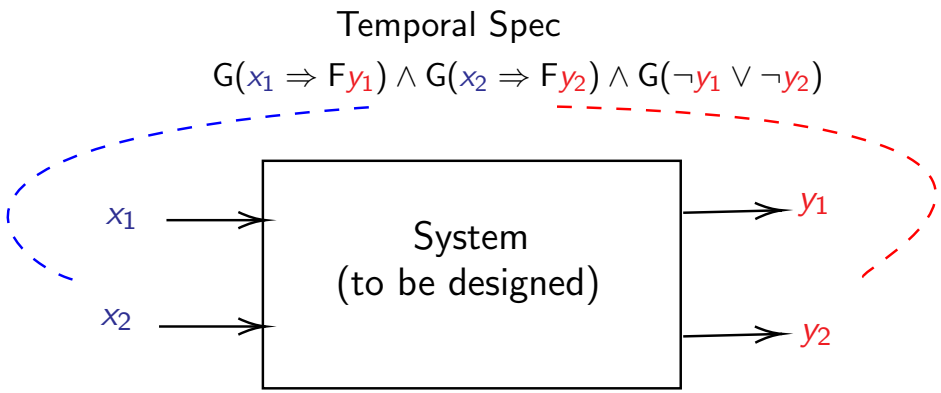

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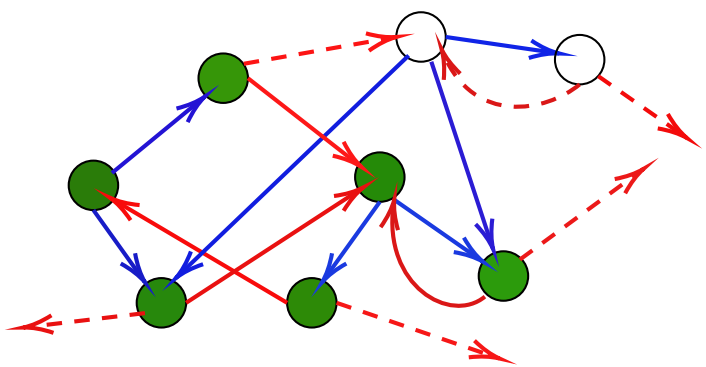

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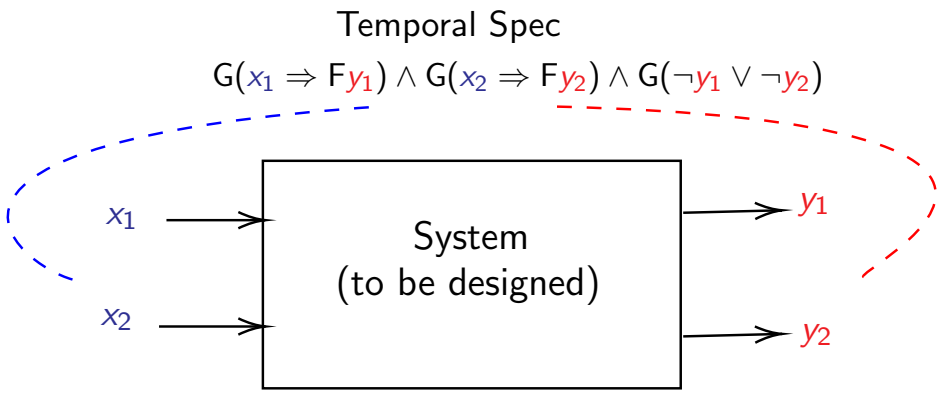

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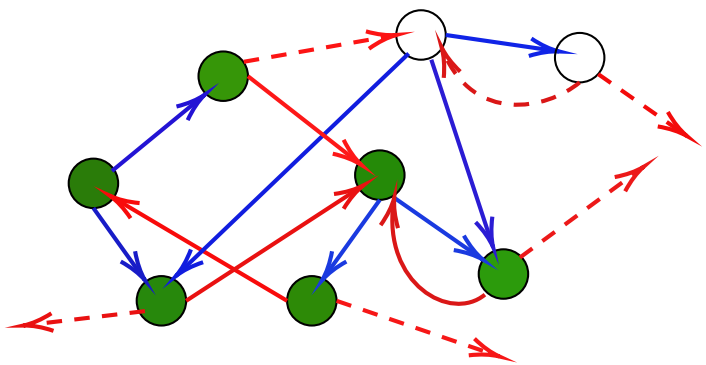

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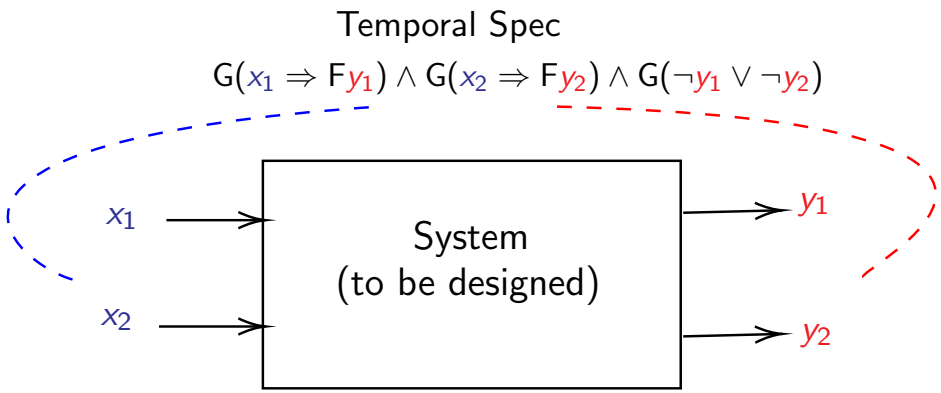

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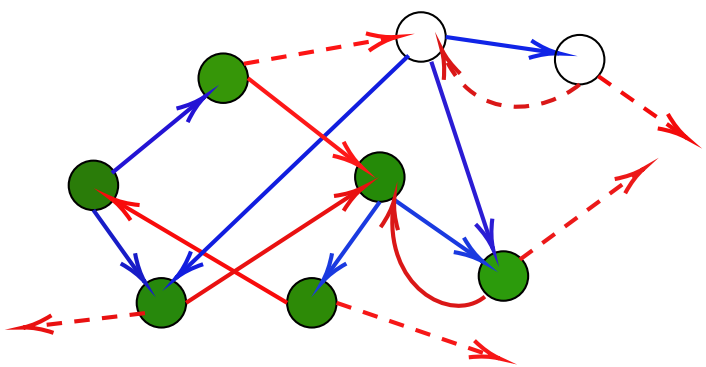

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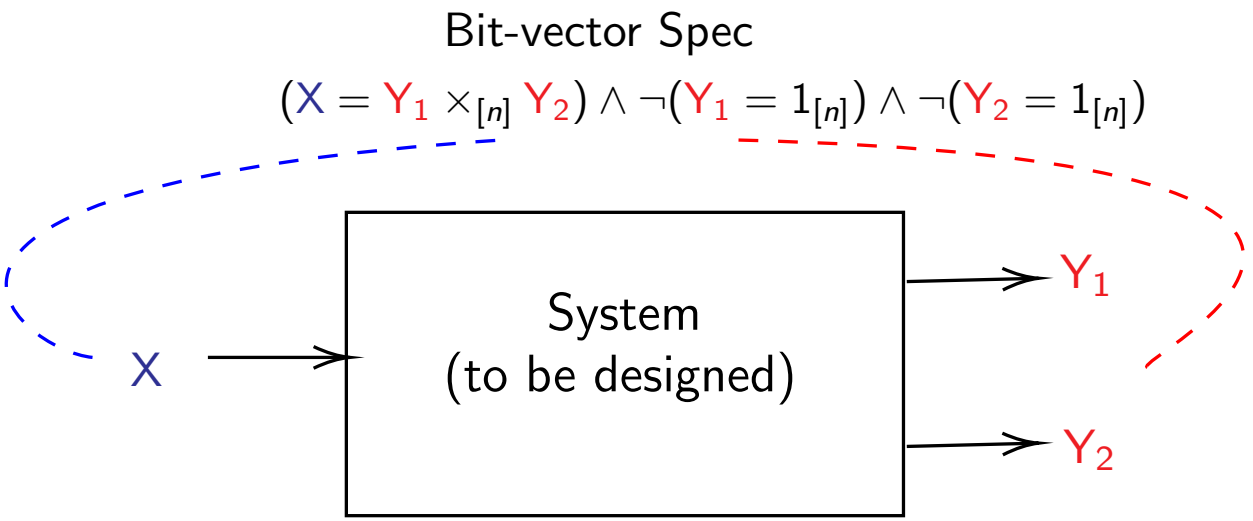

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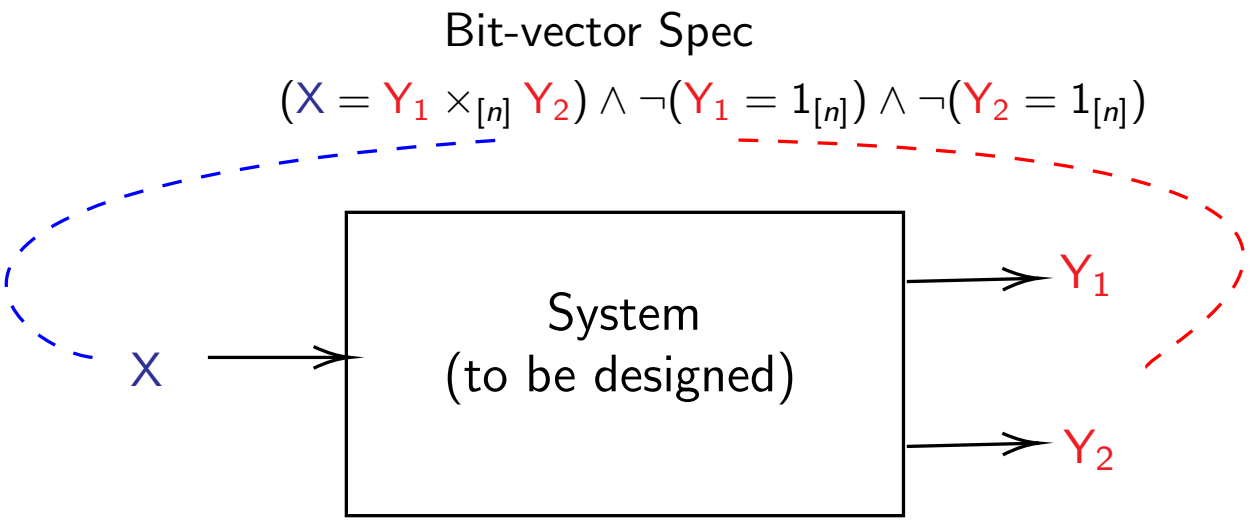

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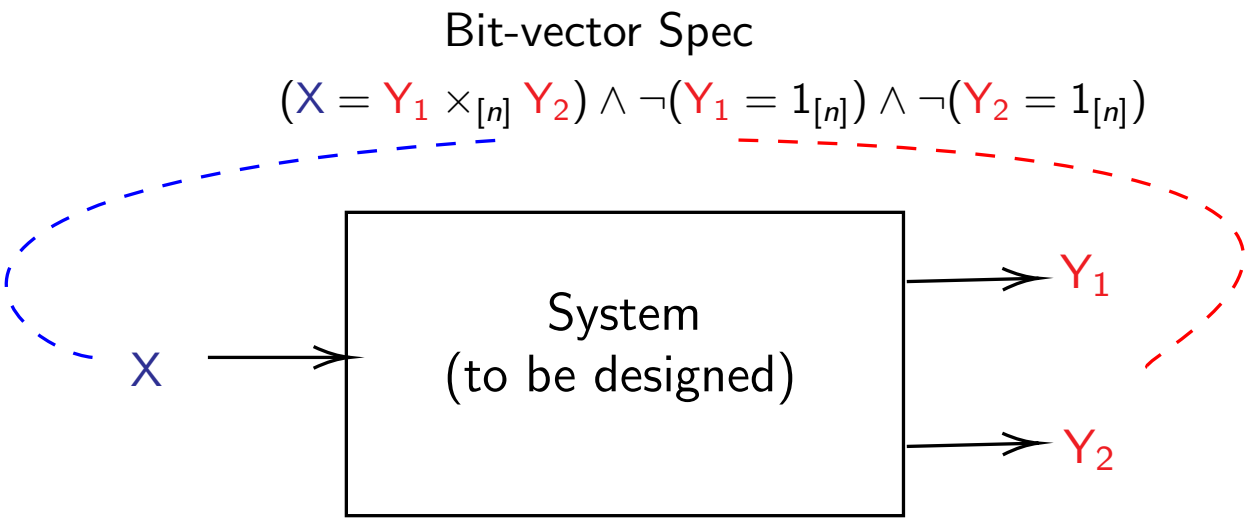

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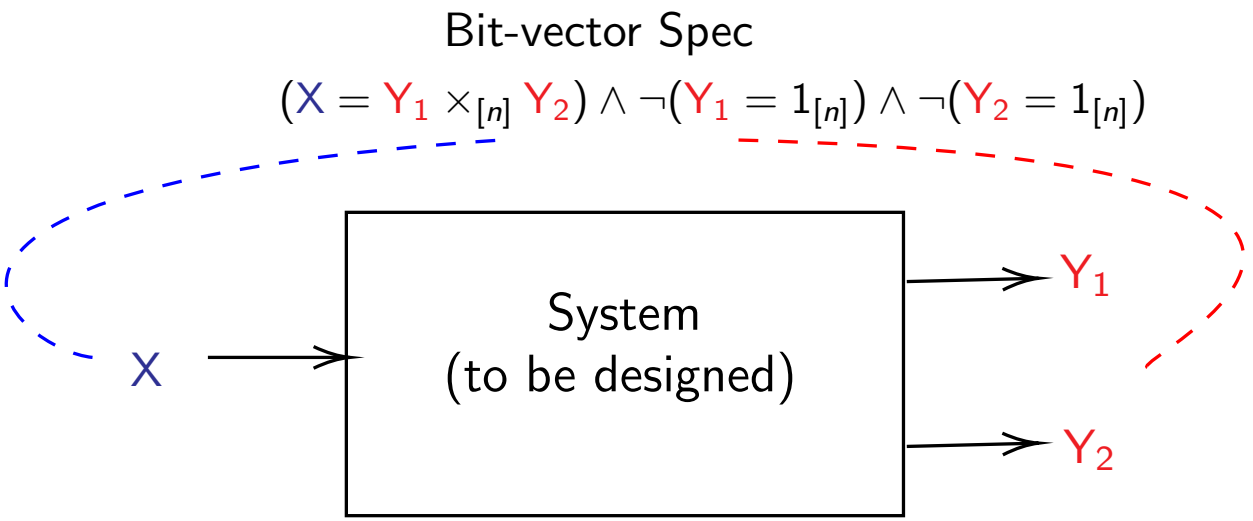

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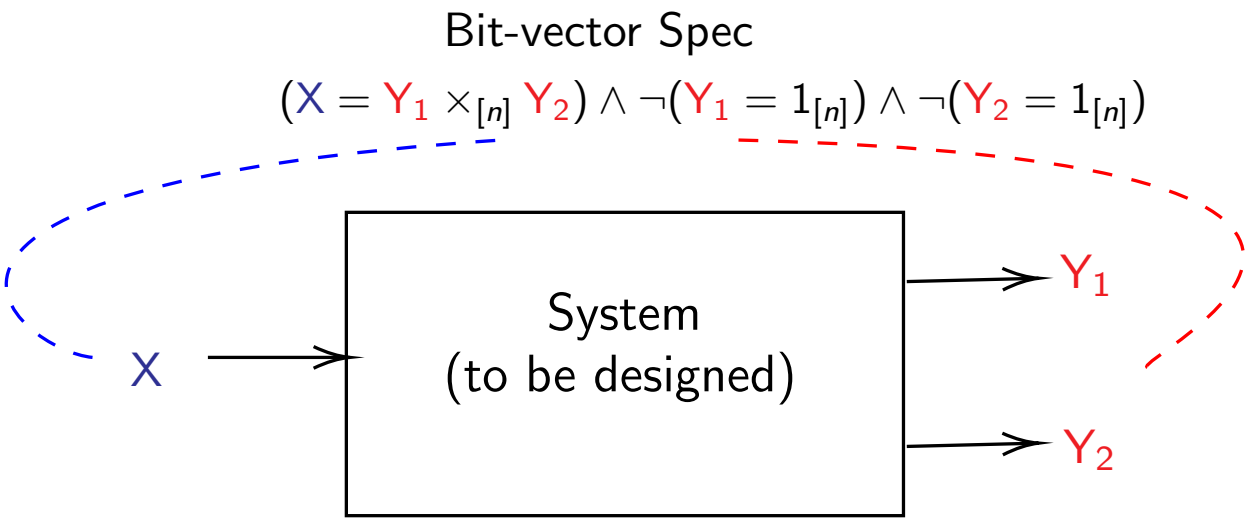

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  - Efficient solution would break crypto systems

Boolean Functional Synthesis

Formal definition

Given Boolean relation $\varphi(x_1, .., x_n, y_1, .., y_m)$

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  - Given X, $\mathrm{F}(\mathrm{X})$, easy to check if $\exists \mathrm{Y}\ \varphi(\mathrm{X}, \mathrm{Y}) = \varphi(\mathrm{X}, \mathrm{F}(\mathrm{X})) = 0$

Applications of Boolean Functional Synthesis

1. Cryptanalysis: Interesting but hard for synthesis!
2. Disjunctive decomposition of symbolic transition relations [Trivedi et al'02]
3. Quantifier elimination, of course!
   - $\exists Y\ \varphi(X, Y) \equiv \varphi(X, F(X))$
4. Certifying QBF-SAT solvers
   - Nice survey of applications by Shukla et al'19
5. Reactive controller synthesis
   - Synthesizing moves to stay within winning region
6. Program synthesis
   - Combinatorial sketching [Solar-Lezama et al'06, Srivastava et al'13]
   - Complete functional synthesis [Kuncak et al'10]
7. Repair/partial synthesis of circuits [Fujita et al'13]

Existing Approaches

1. Closely related to most general Boolean unifiers
   - Boole'1847, Lowenheim'1908, Macii'98
2. Extract Sk. functions from proof of validity of $\forall X \exists Y \varphi(X, Y)$
   - Bendetti'05, Jussilla et al'07, Balabanov et al'12, Heule et al'14
3. Using templates: Solar-Lezama et al'06, Srivastava et al'13
4. Self-substitution + function composition: Jiang'09, Trivedi'03
5. Synthesis from special normal form representation of specification
   - From ROBDDs: Kukula et al'00, Kuncak et al'10, Fried et al'16, Tabajara et al'17
   - From SynNNF: Akshay et al'09
6. Incremental determinization: Rabe et al'17,'18
7. Quantifier instantiation techniques in SMT solvers
   - Barrett et al'15, Bierre et al'17
8. Input/output component separation: C. et al'18
9. Guess/learn Skolem function candidate + check + **repair**
   - John et al'15, Akshay et al'17,'18,'20, Golia et al'20

How Hard (or Easy) Is BFnS?

- Boolean circuit: DAG with AND-, OR-, NOT-labeled nodes
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  - Skolem functions need not be monotone
  - Different argument for lower bounds on Skolem circuits

Some Good and Bad News

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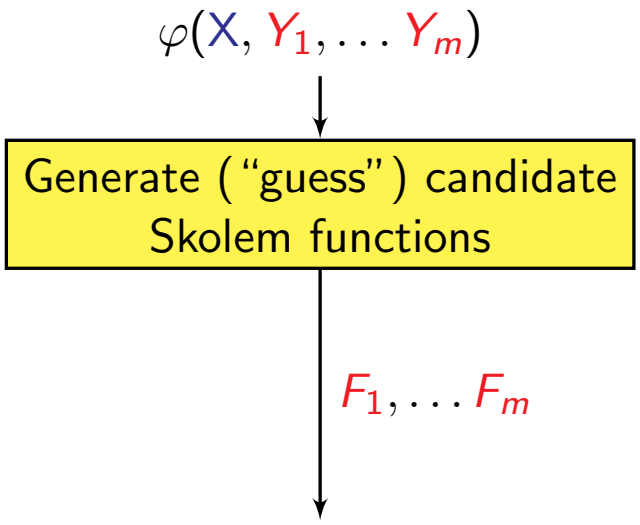
- φ poly-time (size) synthesizable **iff** it is poly-time (size) compilable to Subset And Unrealizable Negation Normal Form (SAUNF)

Guess-Check-Repair: A Useful Paradigm



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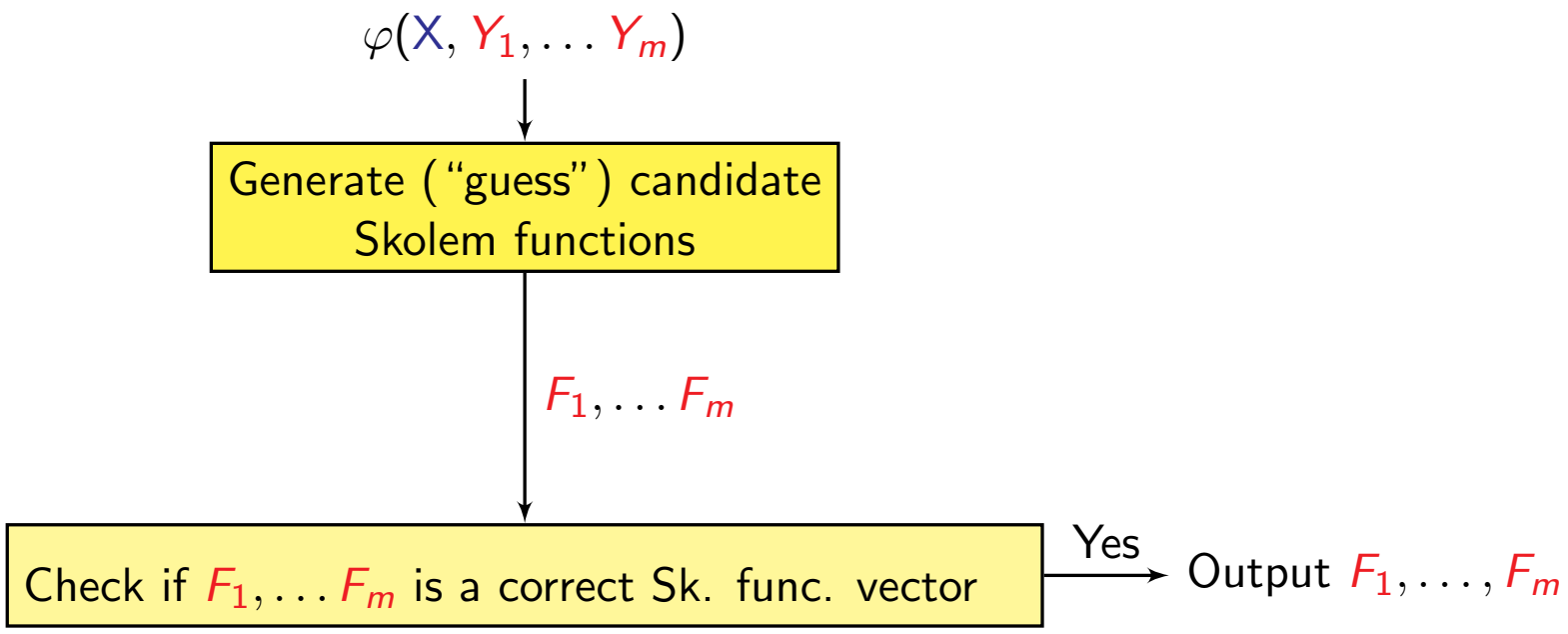

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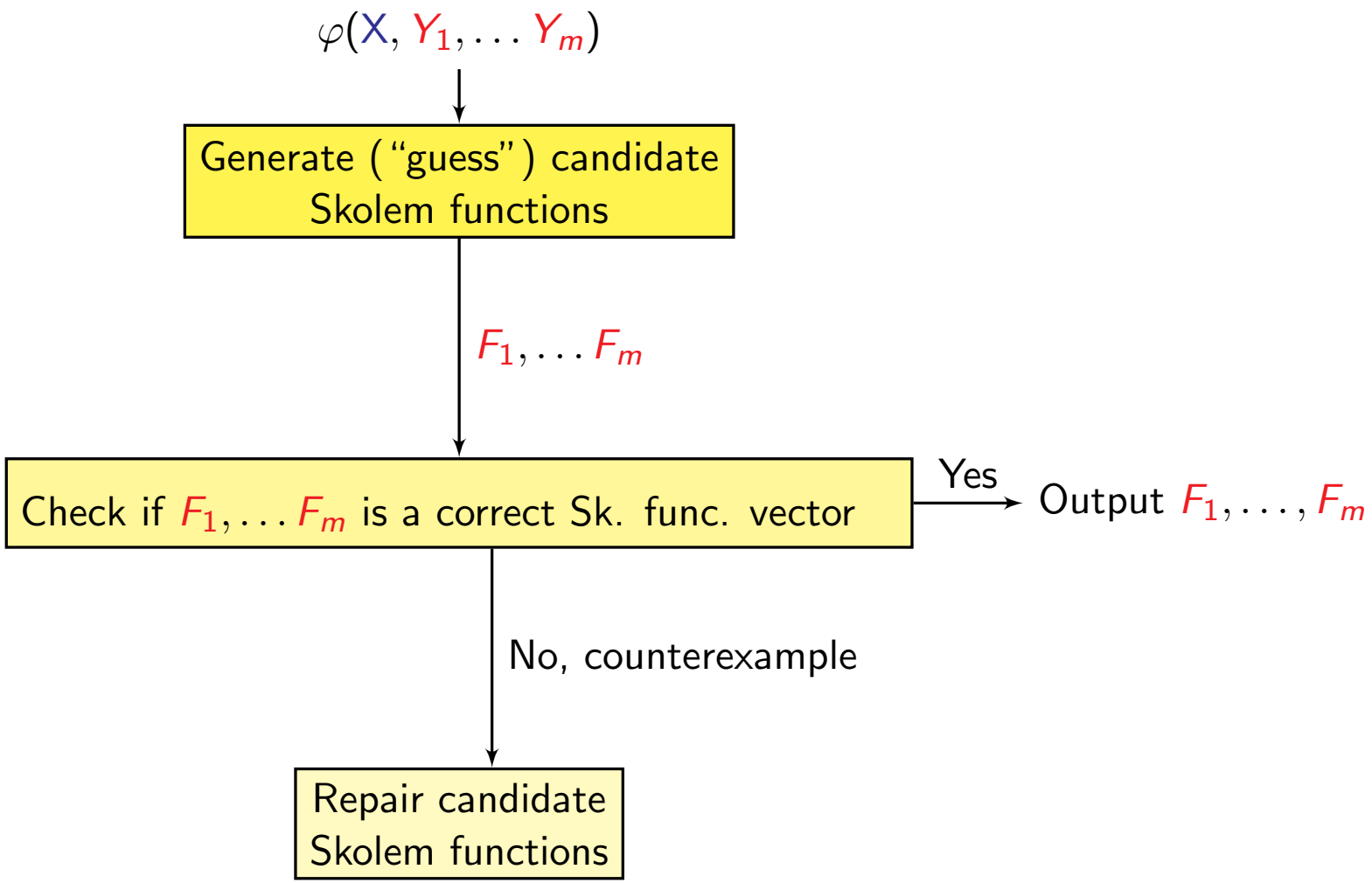

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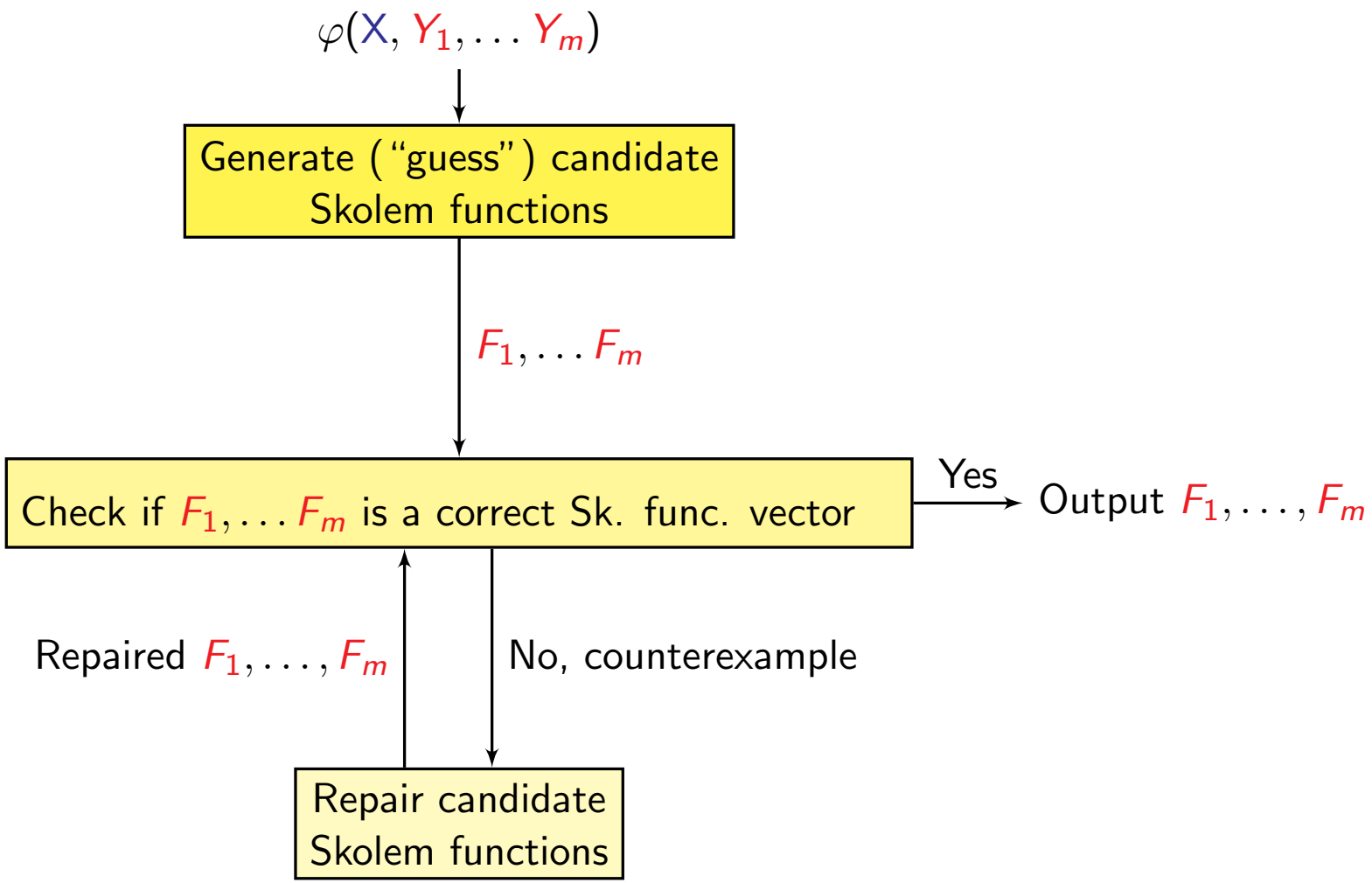

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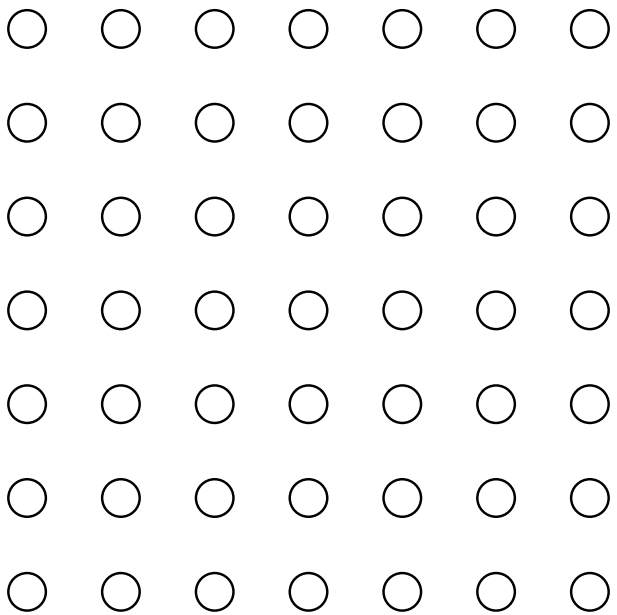

— Set of all valuations of X.

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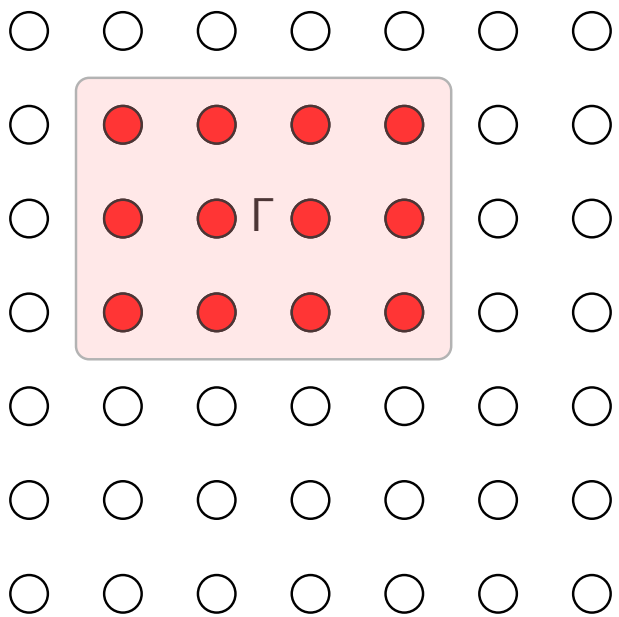

— Can't set y to 1 to satisfy φ: $\Gamma(X) \triangleq \neg\varphi(X, y)[y \mapsto 1]$

E.g. If $\varphi \equiv (x_1 \vee y) \wedge (x_1 \vee x_2 \vee \neg y)$, then
$\Gamma(X) = \neg((x_1 \vee 1) \wedge (x_1 \vee x_2 \vee 0)) = \neg(x_1 \vee x_2) = \neg x_1 \wedge \neg x_2$

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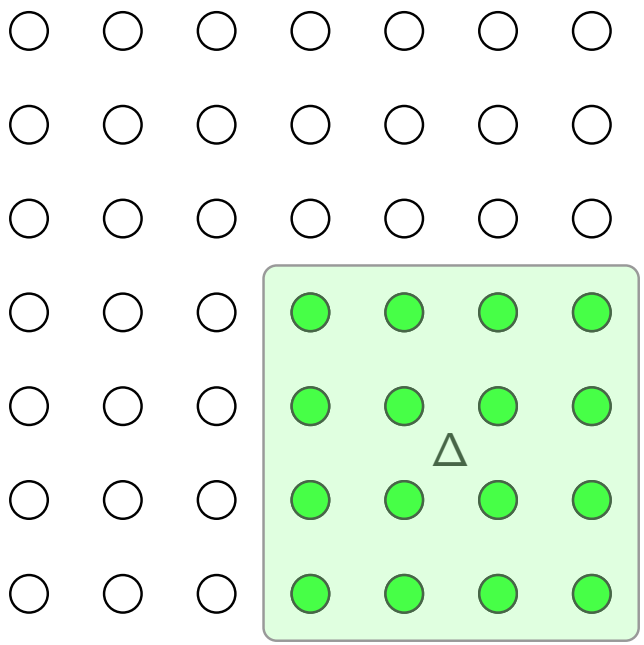

— Can't set y to 0 to satisfy φ: $\Delta(X) \triangleq \neg\varphi(X, y)[y \mapsto 0]$

E.g. If $\varphi \equiv (x_1 \vee y) \wedge (x_1 \vee x_2 \vee \neg y)$, then
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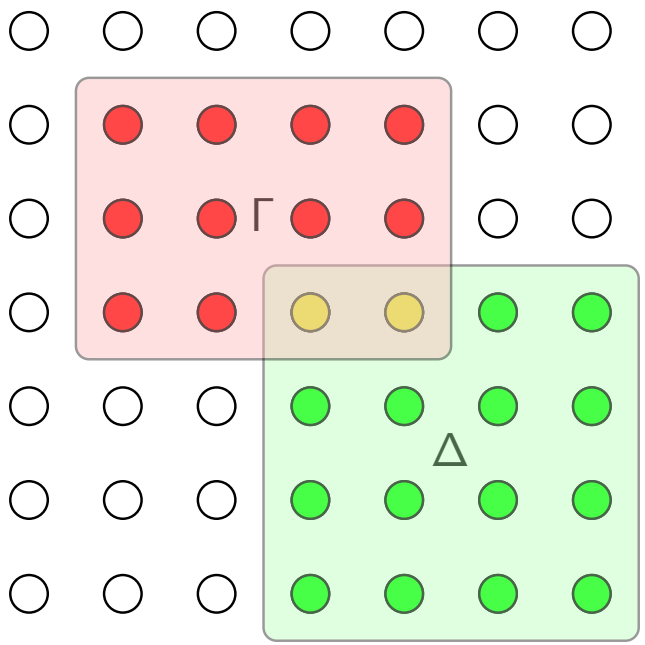

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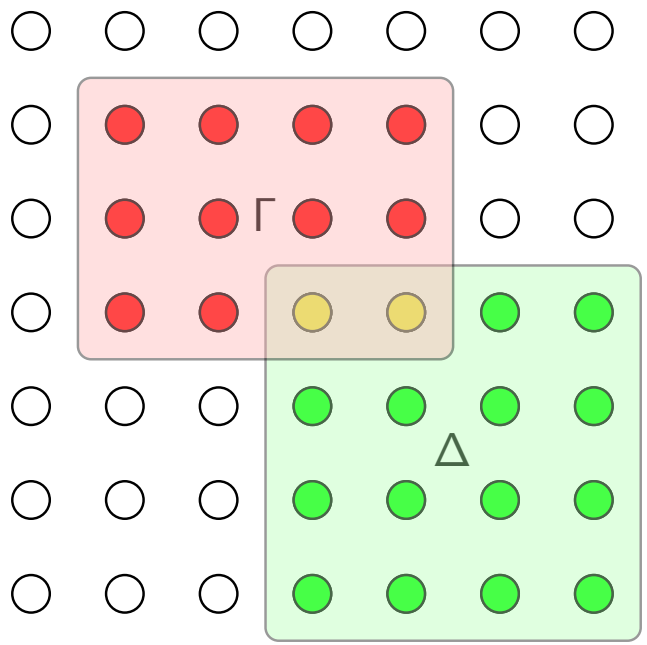

Lemma [Trivedi'03, Jiang'09,Fried et al'16]

Every Skolem function for y in φ must

- Evaluate to 1 in $(\Delta \setminus \Gamma)$ and to 0 in $(\Gamma \setminus \Delta)$
- Be an **interpolant** of $(\Delta \setminus \Gamma)$ and $(\Gamma \setminus \Delta)$

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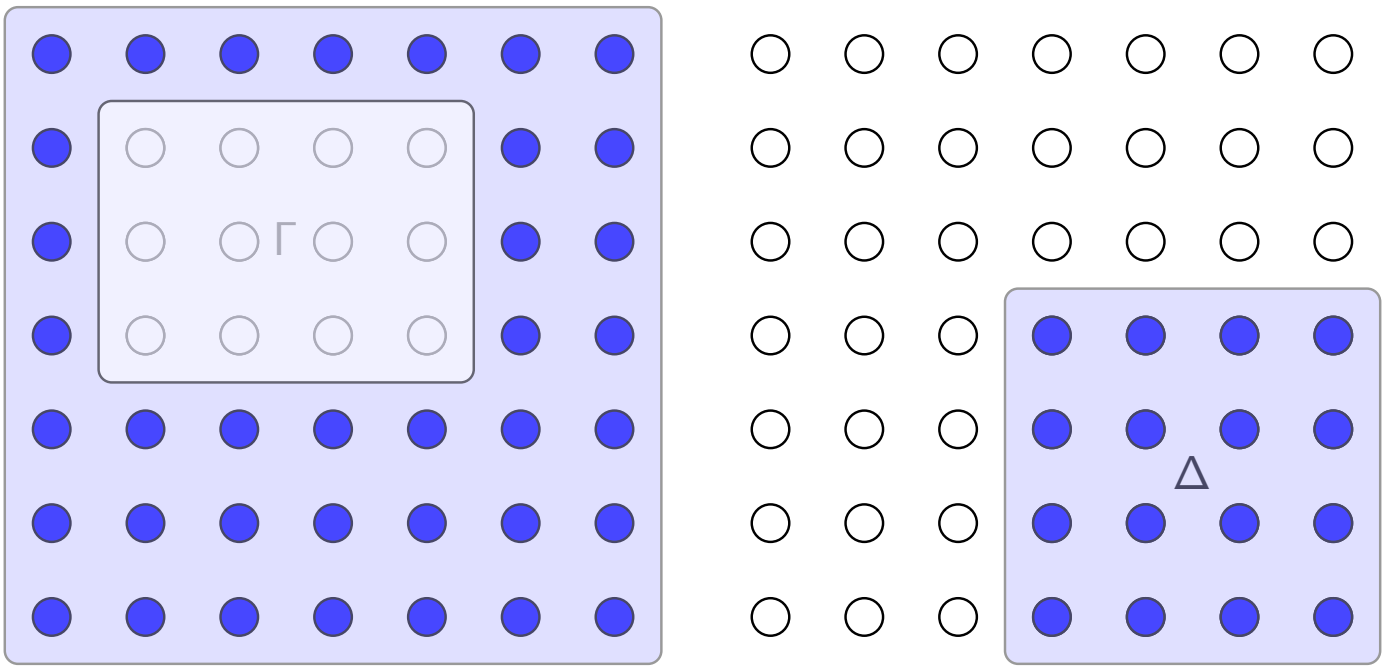

— Specific interpolants of $(\Delta \setminus \Gamma)$ & $(\Gamma \setminus \Delta)$

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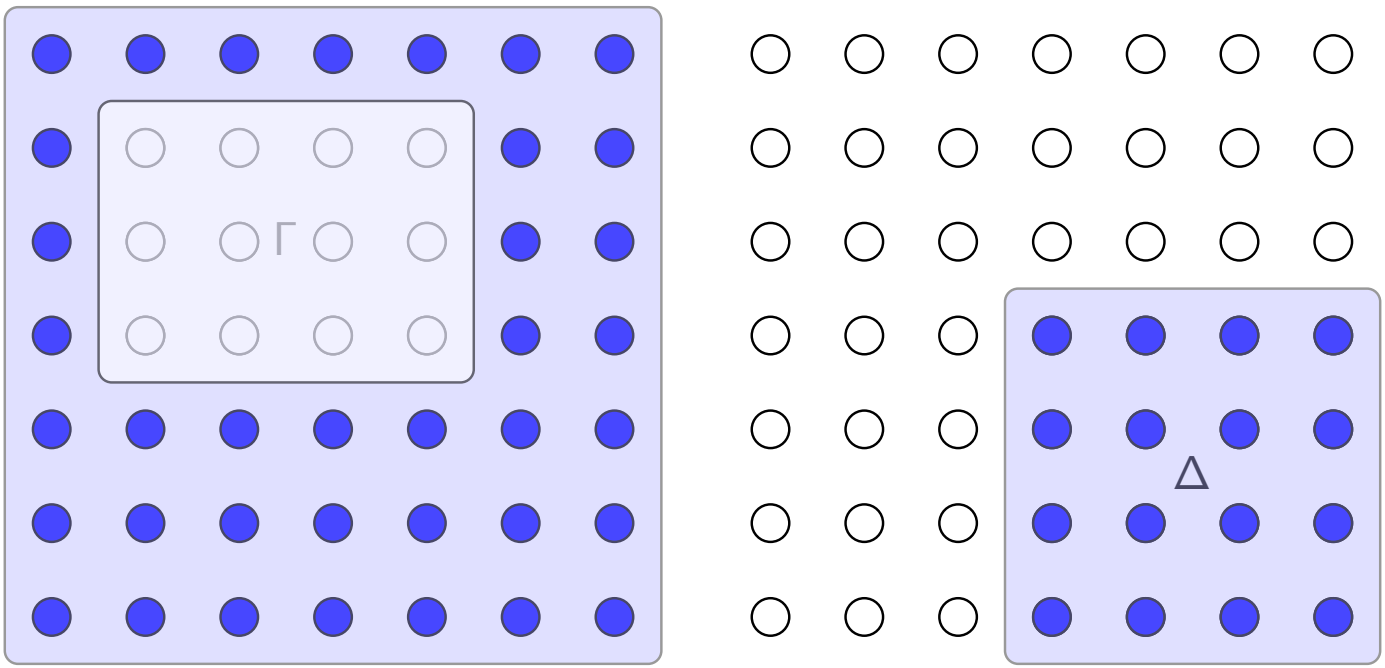

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- $\neg\Gamma \triangleq \varphi(X, y)[y \mapsto 1] \equiv \varphi(X, 1)$: Easy solution for 1 output var
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Drawbacks of approach:

- Existential quant elimination over long sequences of outputs expensive
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Can we work around these drawbacks?

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A $|X|$-input, $|Y|$-output circuit computing the desired Skolem function vector $(F_1, \ldots F_m)$ can be constructed with

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Sufficient to compute the G_i functions

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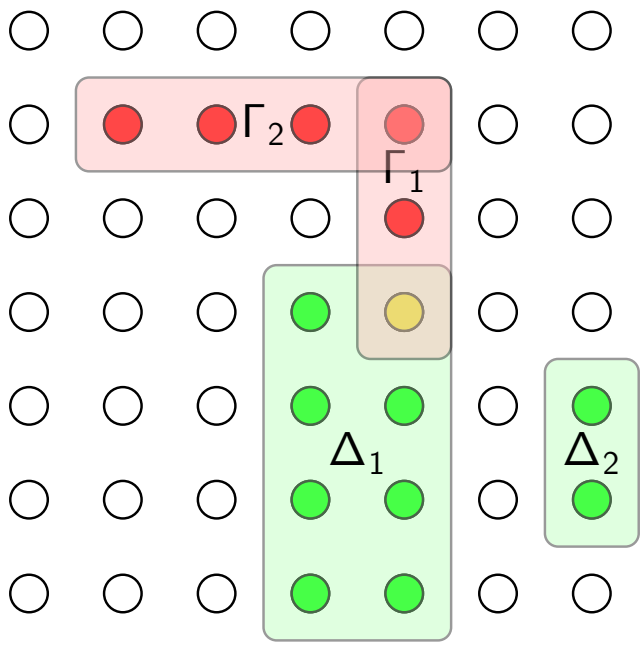

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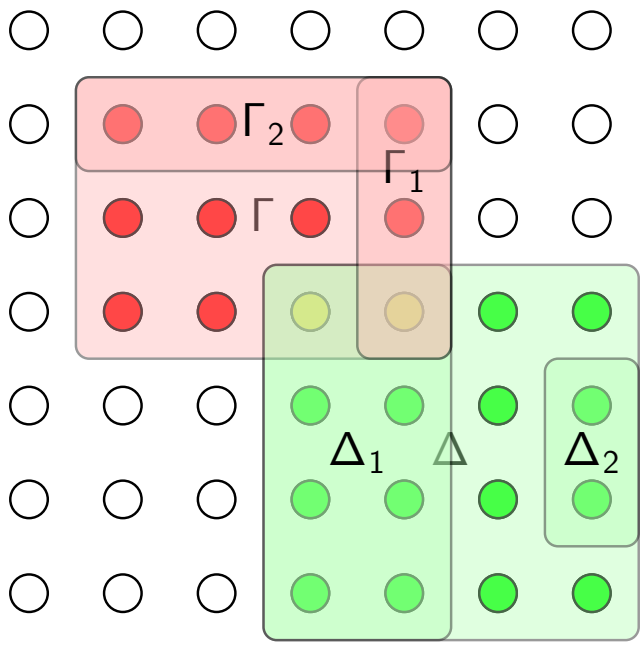

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- Using under-approximations of $\Gamma_1^{y_i}$ and $\Delta_1^{y_i}$ yields under-approximations of Γ^{y_i} and Δ^{y_i}

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If $\Gamma^{y_1} \triangleq \neg\exists y_2 \ldots y_m\ (\varphi_1 \vee \varphi_2)(X, 1, \ldots)$, then $\Gamma_1^{y_1} \wedge \Gamma_2^{y_1} \Leftrightarrow \Gamma^{y_1}$

If $\Delta^{y_1} \triangleq \neg\exists y_2 \ldots y_m\ (\varphi_1 \vee \varphi_2)(X, 0, \ldots)$, then $\Delta_1^{y_1} \wedge \Delta_2^{y_1} \Leftrightarrow \Delta^{y_1}$

What if calculating $\Gamma_1^{y_i}$ or $\Delta_1^{y_i}$ hard?

- Long sequences of quantification are of concern!
- Using under-approximations of $\Gamma_1^{y_i}$ and $\Delta_1^{y_i}$ yields under-approximations of Γ^{y_i} and Δ^{y_i}
  - Not so for over-approximations!
    - ${\Gamma_1}^{y_i} \vee (\wedge)\ {\Gamma_2}^{y_i} \Rightarrow (\Leftrightarrow)\Gamma^{y_i}$
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"Guess"-ing Compositionally: A High-level View

Suppose $\varphi(X, Y) \equiv \varphi_1(X, Y) \vee \varphi_2(X, Y)$, where $Y = y_1, \ldots y_m$

$\Gamma_1^{y_1} \triangleq \neg\exists y_2 \ldots y_m\ \varphi_1(X, 1, y_2 \ldots y_m)$ $\quad \Delta_1^{y_1} \triangleq \neg\exists y_2 \ldots y_m\ \varphi_1(X, 0, \ldots)$

$\Gamma_2^{y_1} \triangleq \neg\exists y_2 \ldots y_m\ \varphi_2(X, 1, \ldots)$ $\quad \Delta_2^{y_1} \triangleq \neg\exists y_2 \ldots y_m \varphi_2(X, 0, \ldots)$

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- Fortunately, non-trivial under-approx of Γ^{y_i} and Δ^{y_i} not hard to obtain

"Guess"-ing with under-approximations of Γ, Δ

- Suppose $\gamma_1^{y_i} \Rightarrow \Gamma_1^{y_i}$; $\delta_1^{y_i} \Rightarrow \Delta_1^{y_i}$

"Guess"-ing with under-approximations of Γ, Δ

- Suppose $\gamma_1^{y_i} \Rightarrow \Gamma_1^{y_i}$; $\delta_1^{y_i} \Rightarrow \Delta_1^{y_i}$
- $\varphi \equiv \varphi_1 \wedge \varphi_2$
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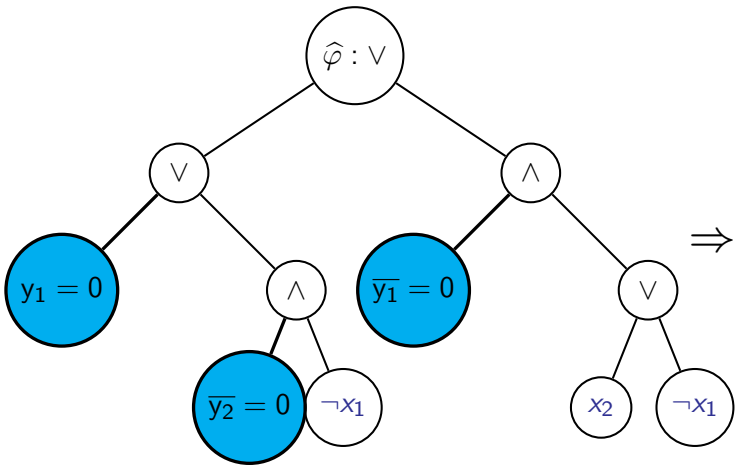
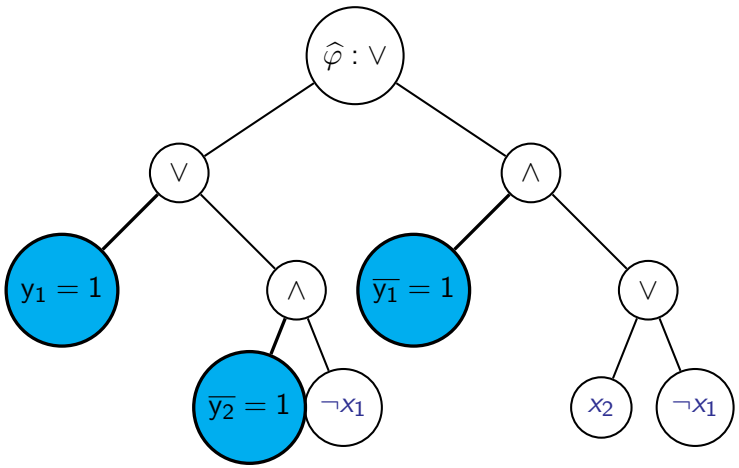
  - $\gamma_1^{y_i} \wedge \gamma_1^{y_i} \Rightarrow \Gamma_1^{y_i} \wedge \Gamma_1^{y_i} \Leftrightarrow \Gamma^{y_i}$
- Similarly for Δ^{y_i}

Illustrating Approximations



$\Rightarrow \exists y_1 y_2 \varphi \Rightarrow$



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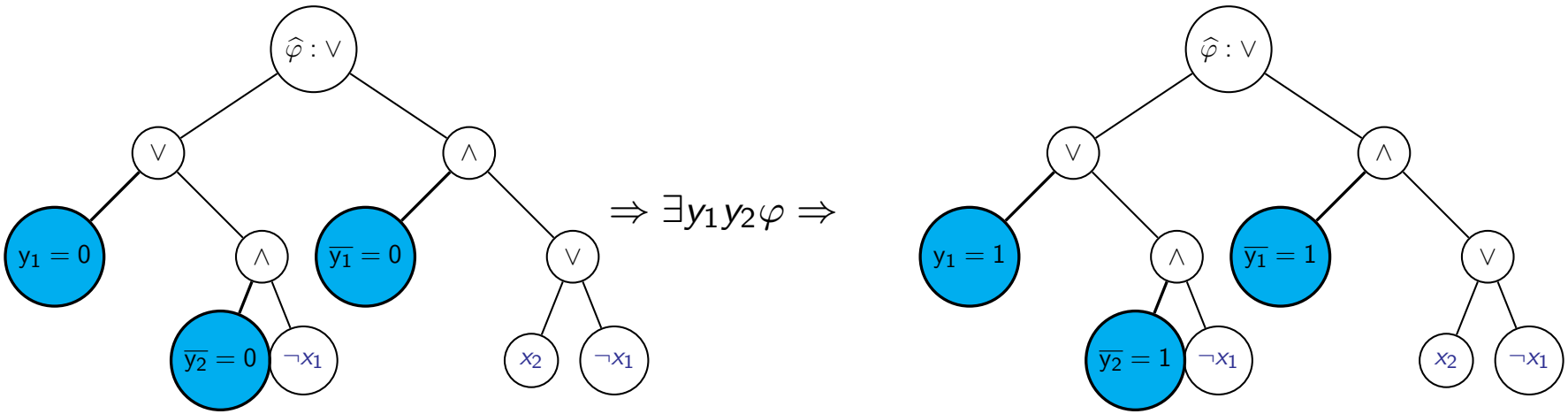

- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{0..0}^{i}, y_{i+1} \ldots y_m, \overbrace{0..0}^{i}, \neg y_{i+1} \ldots \neg y_m) \Rightarrow \exists y_1 \ldots y_i \ \varphi(\ldots)$
- $\widehat{\varphi}(x_1 \ldots x_n, \overbrace{1..1}^{i}, y_{i+1} \ldots y_m, \overbrace{1..1}^{i}, \neg y_{i+1} \ldots \neg y_m) \Leftarrow \exists y_1 \ldots y_i \ \varphi(\ldots)$

Checking correctness of "guess"-ed Skolem functions

Given candidate Skolem functions $F_1, \ldots F_m$,

$$\text{Is } \forall X(\exists Y \varphi(X, Y) \Leftrightarrow \varphi(X, F(X))) ?$$

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- Propositional error formula $\varepsilon(X, Y, Y')$:

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  - $\varphi(\sigma[X], \sigma[Y'])] = 1, \quad \sigma[Y] = F(\sigma[X]), \quad \varphi(\sigma[X], \sigma[Y]) = 0$

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- Suppose σ: satisfying assignment of ε
  - $\varphi(\sigma[X],\sigma[Y'])=1, \quad \sigma[Y]=F(\sigma[X]), \quad \varphi(\sigma[X],\sigma[Y])=0$
  - σ is **counterexample** to the claim that $F_1, \ldots F_m$ is a correct Skolem function vector

Counterexample Generalization

Recall: Skolem functions guessed from approximations of $\exists y_{i+1} \ldots \exists y_m \, \varphi(X, y_1, \ldots y_{i-1}, y_i, y_{i+1}, \ldots y_m)$

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- Let $\exists y_{i+1} \ldots \exists y_m \, \varphi(X, Y) \Rightarrow \Theta_i(X, y_1, \ldots y_{i-1}, y_i)$ Over-approx

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Under-approximations

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Must ensure that $(X, G_1, \ldots G_{j-1})$ never evaluates to π

Repairing "guess"-ed candidate Skolem functions

- Every model of $\mu(X, y_1, \ldots y_{j-1})$ gives a problematic combination of $G_1, \ldots G_{j-1}$ values

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    - If G_{j-1} is $\neg\gamma_{j-1}$, $\quad \gamma_{j-1} \leftarrow \gamma_{j-1} \vee \mu|_{\sigma[y_{j-1}]}$
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Expansion-based repair

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- Flip G_{j-1} whenever μ holds
  - Recall $G_{j-1} \in \{\neg\gamma_{j-1}, \delta_{j-1}\}$
  - **Only** source of error: under-approximation of $\neg\exists y_j, \ldots \exists y_m\, \varphi(X, y_1, \ldots y_{j-2}, y_{j-1}, y_j, \ldots y_m)$
  - Repair: **Expand under-approximation**
    - If G_{j-1} is $\neg\gamma_{j-1}$, $\quad \gamma_{j-1} \leftarrow \gamma_{j-1} \vee \mu|_{\sigma[y_{j-1}]}$
    - If G_{j-1} is δ_{j-1}, $\quad \delta_{j-1} \leftarrow \delta_{j-1} \vee \mu|_{\sigma[y_{j-1}]}$

Counter-example guided repair by expanding δ_i's and γ_i's.

Expansion-based repair

Simple argument for termination – expansions can't go on forever

Counterexample-guided repair

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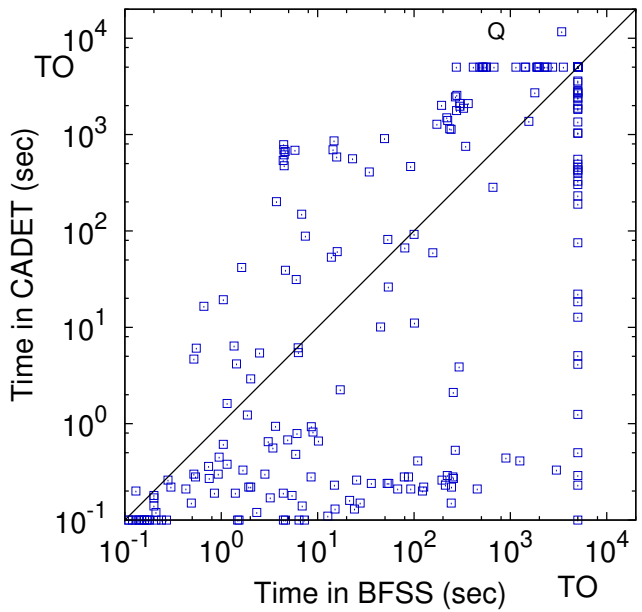
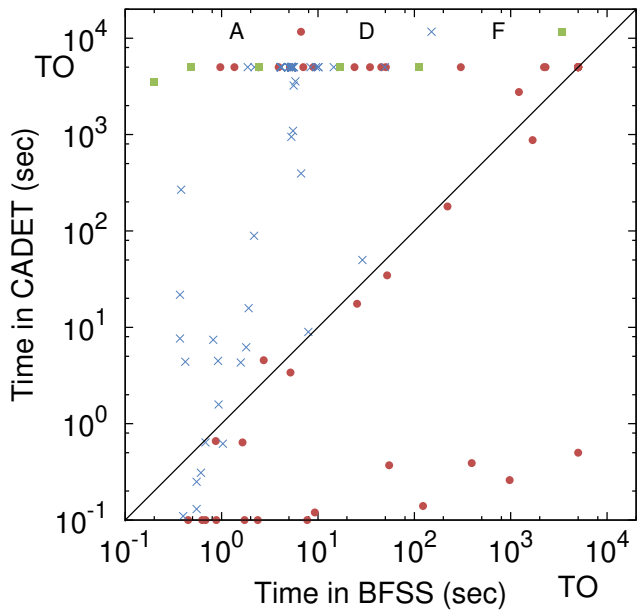
- Can repair multiple Skolem functions starting from one counterexample

Experimental Comparison

BFSS (Cex-guided-repair) vis-a-vis CADET (incremental determinization) [Rabe & Seshia'16]

[Comparisons with other tools in FMSD 2020 paper]





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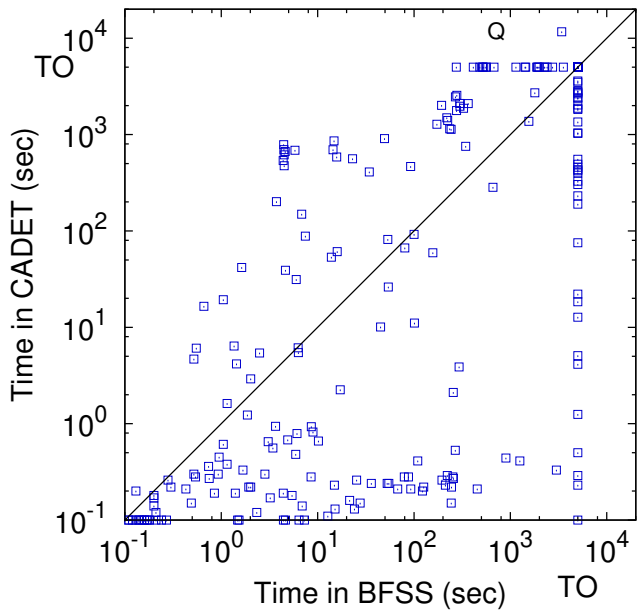

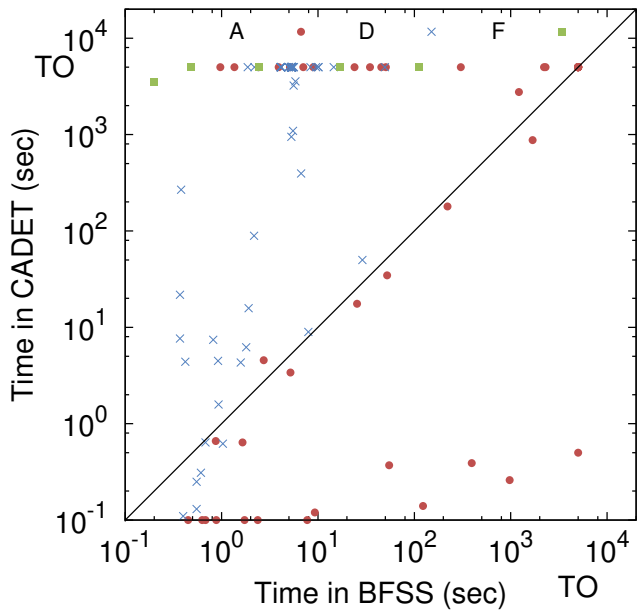

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- Mixed results: tools have orthogonal strengths
- Using CADET and BFSS as a portfolio solver sounds promising

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Manthan: Golia et al [GMR'20]

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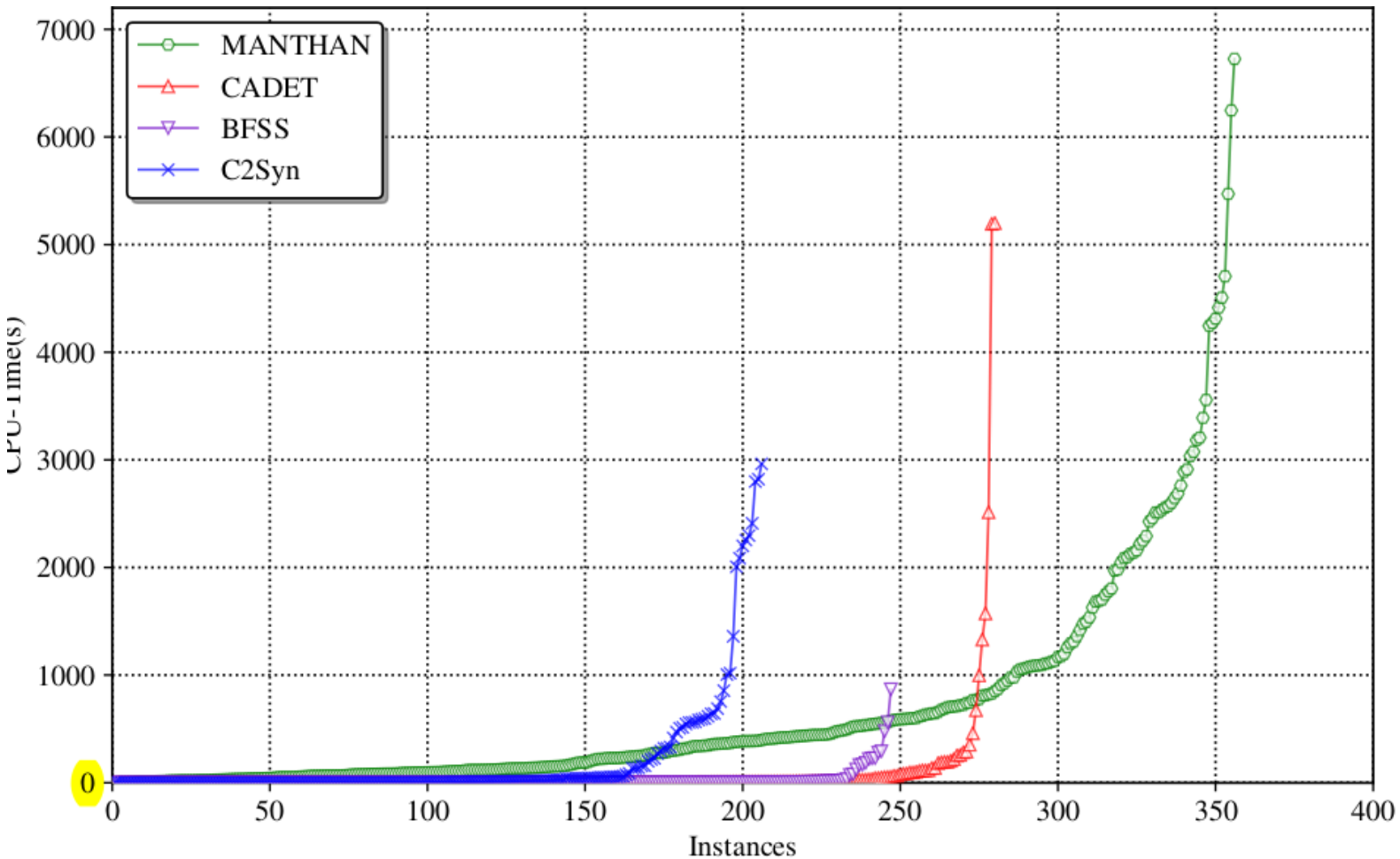
  - Update
    - F_i to $F_i \vee \beta$ if $\sigma[y_i] = 0$
    - F_i to $F_i \wedge \neg\beta$ if $\sigma[y_i] = 1$

Some more experimental comparisons [GMR20]



[Courtesy "Manthan: A Data Driven Approach to Boolean Functional Synthesis", Golia et al, CAV 2020]

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- Recent results (BFSS, Manthan) extremely promising!!!