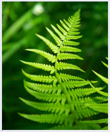
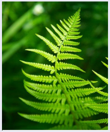


A Finitary Analogue of the Downward Löwenheim-Skolem Property



Abhisekh Sankaran
IMSc, Chennai

Formal Methods Update Meet
IIT Mandi
July 18, 2017

Introduction

- The Downward Löwenheim-Skolem theorem (DLS) is amongst the earliest results in classical model theory.
- The first version of DLS is by Löwenheim in his paper Über Möglichkeiten im Relativkalkül (1915) and reads as follows:

  If a first order sentence over a countable vocabulary has an infinite model, then it has a countable model.
- Historically,
  - 1915: First version of DLS by Löwenheim. His proof used Konig's lemma (1927) without proving it.
  - 1920s: First fully self-contained proof of Löwenheim's statement and various generalizations of DLS by Skolem
  - 1936: The most general version of DLS by Mal'tsev
- DLS + compactness = first order logic (Lindström 1969).

Outline of the talk

A. Notions:

- The Downward Löwenheim-Skolem Property: DLSP
- The Equivalent Bounded Substructure Property: EBSP

B. Results:

- Classes of finite structures satisfying EBSP
- Closure properties of EBSP
- Techniques and f.p.t. algorithms
- Connection with Nature

Some assumptions and notation for the talk

Assumptions:

- First order (FO) logic
- Finite relational vocabularies (i.e. only predicates)

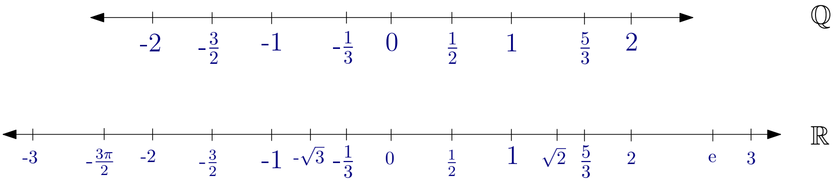
Notation:

- $\mathcal{A} \subseteq \mathcal{B}$ means $\mathcal{A}$ is a substructure of $\mathcal{B}$.
- $\mathsf{U}_{\mathcal{A}}$ denotes universe of $\mathcal{A}$.
- $|\mathcal{A}|$ denotes size of $\mathsf{U}_{\mathcal{A}}$.

A. Notions

The Downward Löwenheim-Skolem Property

FO-similarity of structures



$\mathbb{Q}$ and $\mathbb{R}$ are FO-similar

We say structures $\mathcal{A}$ and $\mathcal{B}$ are FO-similar, denoted $\mathcal{A} \equiv \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ agree on all properties that can be expressed in FO.

The Downward Löwenheim-Skolem Property

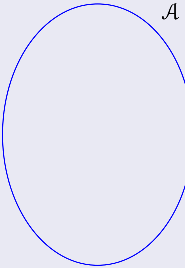
Definition

We say DLSP holds if

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if

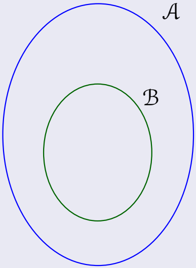


$\forall \mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if

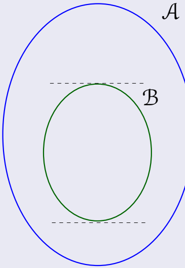


$\forall \mathcal{A}$
$\exists \mathcal{B} \subseteq \mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

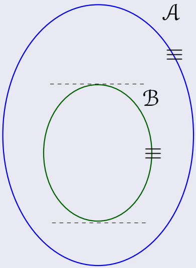
We say DLSP holds if



$\forall\, \mathcal{A}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$
(i) the size of $\mathcal{B}$ is $\leq \omega$

Definition

We say DLSP holds if



$\forall\, \mathcal{A}$

$\exists\, \mathcal{B} \subseteq \mathcal{A}$

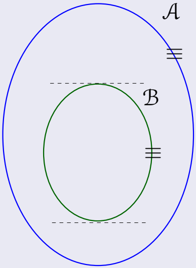
(i) the size of $\mathcal{B}$ is $\leq \omega$

(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

The Downward Löwenheim-Skolem Property

Definition

We say DLSP holds if



$\forall \mathcal{A}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) the size of $\mathcal{B}$ is $\leq \omega$
(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

"$\mathcal{A}$ has an FO-similar substructure of size $\leq \omega$"

The Downward Löwenheim-Skolem theorem

Theorem (Löwenheim 1915, Skolem 1920s)

DLSP is true over all structures.

Downward Löwenheim-Skolem theorem in the finite

- Does not make sense when taken as is.
- No recursive version of Löwenheim's statement – there is no recursive function bounding the size of a small model of an FO sentence.
- Grohe showed a stronger negative result:
  For every recursive function $f : \mathbb{N} \to \mathbb{N}$, there is an FO sentence φ and $n \geq f(|\varphi|)$, such that φ has a model of each size $\geq n$ but no model of size $< n$.
- Quoting Grohe, the above counterexample "refutes almost all possible extensions of the classical Löwenheim-Skolem theorem to finite structures".

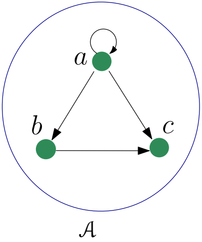
Classical theorems over classes of finite structures

- Most theorems from classical model theory fail over all finite structures (DLS, preservation theorems, interpolation theorems, etc.)
- Active research in last 15 years to "recover" classical theorems over classes interesting from structural and algorithmic perspectives.
- Acyclic, bounded degree, wide, bounded tree-width – Łoś-Tarski pres. theorem
- In addition to the above, quasi-wide classes, classes excluding atleast one minor – homomorphism pres. theorem
- No such studies in the literature for the DLS theorem.

A new logic based combinatorial property of finite structures

FO over finite structures

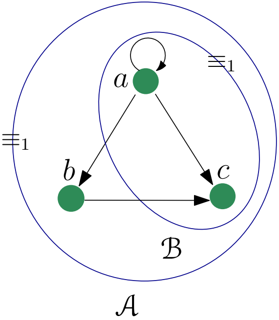
1. Any finite structure can be captured upto isomorphism by FO.



$$\begin{aligned}\varphi := \exists x\, \exists y\, \exists z\, \forall w \\ \Big(& (x \neq y \wedge y \neq z \wedge z \neq x) \bigwedge \\ & (w = x \vee w = y \vee w = z) \bigwedge \\ & (E(x,x) \wedge E(x,y) \wedge E(x,z) \wedge E(y,z)) \bigwedge \\ & \neg(E(y,y) \vee E(z,z) \vee E(y,x) \vee E(z,y) \vee E(z,x)) \Big)\end{aligned}$$

2. Then FO-similarity $=$ isomorphism in the finite.
3. Define a weaker version of FO-similarity by considering FO sentences of a fixed quantifier nesting depth.

m-similarity of structures



$\mathcal{A}$ and $\mathcal{B}$ are 1-similar, but not 2-similar.

We say structures $\mathcal{A}$ and $\mathcal{B}$ are m-similar, denoted $\mathcal{A} \equiv_m \mathcal{B}$, if $\mathcal{A}$ and $\mathcal{B}$ agree on all properties that can be expressed using FO sentences having quantifier nesting depth m.

The Equivalent Bounded Substructure Property

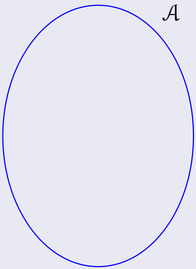
Definition

We say EBSP holds if

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if

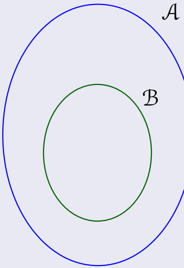


$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if

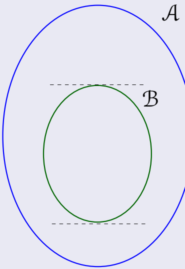


$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if

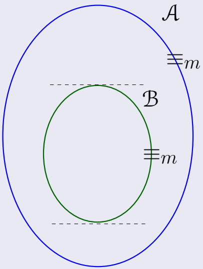


$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}|$ is bounded in m

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if



$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$

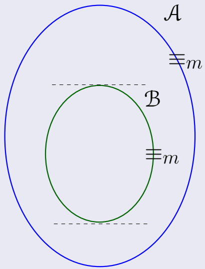
(i) $|\mathcal{B}|$ is bounded in m

(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if



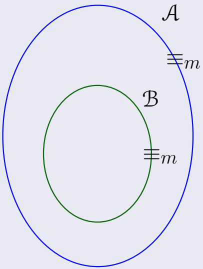
$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}|$ is bounded in m
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if there exists a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that



$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$

$\exists \mathcal{B} \subseteq \mathcal{A}$

(i) $|\mathcal{B}| \leq \theta(m)$

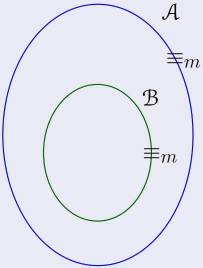
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

We say EBSP holds if there exists a witness function $\theta : \mathbb{N} \rightarrow \mathbb{N}$ such that



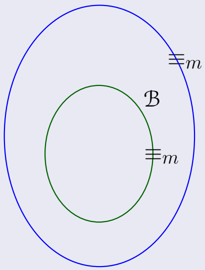
$\forall \mathcal{A} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

"$\mathcal{A}$ has a small m-similar substructure"

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of finite structures, we say EBSP($\mathcal{S}$) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

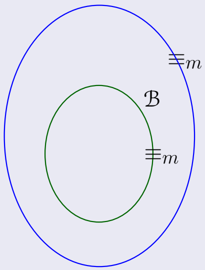


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

The Equivalent Bounded Substructure Property

Definition

Given a class $\mathcal{S}$ of finite structures, we say EBSP($\mathcal{S}$) holds if there is a witness function $\theta : \mathbb{N} \to \mathbb{N}$ such that

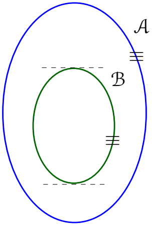


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
$\exists \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$
(i) $|\mathcal{B}| \leq \theta(m)$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

" $\mathcal{A}$ has a small m-similar substructure" – over $\mathcal{S}$

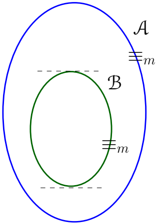
EBSP($\mathcal{S}$) as a finitary analogue of DLSP

<u>DLSP</u>



$\forall\, \mathcal{A}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}$
(i) $|\mathcal{B}| \leq \omega$
(ii) $\mathcal{B}$ is FO-similar to $\mathcal{A}$

<u>EBSP($\mathcal{S}$) for a fixed m</u>



Let $p = \theta(m)$

$\forall\, \mathcal{A} \in \mathcal{S}$
$\exists\, \mathcal{B} \subseteq \mathcal{A}, \quad \mathcal{B} \in \mathcal{S}$
(i) $|\mathcal{B}| \leq p$
(ii) $\mathcal{B}$ is m-similar to $\mathcal{A}$

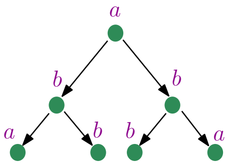
B. Results

Classes that satisfy EBSP

Posets satisfying EBSP

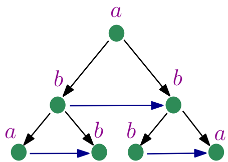
Words and trees (unordered, ordered, ranked)

- Classically studied structures



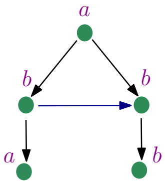
Unordered Σ-tree

$\Sigma = \{a, b\}$



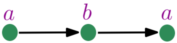
Ordered Σ-tree

$\Sigma = \{a, b\}$



Ordered Σ-tree ranked by ρ

$\Sigma = \{a, b\}$; $\rho = \{a \to 2, b \to 1)\}$



Σ-word

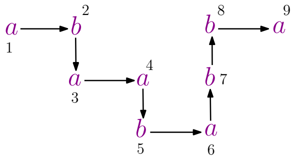
$\Sigma = \{0, 1\}$

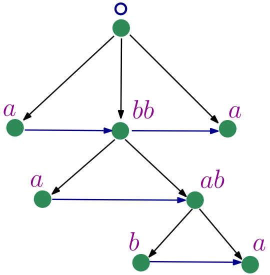
Nested words

- Introduced by Alur and Madhusudan in 2004 as joint generalization of words and ordered unranked trees.

$\mathsf{W} = (abaababba, \rightsquigarrow)$

$\rightsquigarrow = \{(2,8),(4,7)\}$





Regular languages of words, trees and nested words

- A regular language of words/trees/nested words is a class of words/trees/nested words that can be recognized by a finite word/tree/nested word automaton.
- Recall: EBSP($\mathcal{S}$) says for each m, that a large $\mathcal{S}$-structure contains a small m-similar $\mathcal{S}$-substructure.

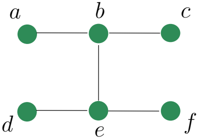
Theorem

Let $\mathcal{S}$ be a regular language of words, trees (unordered, ordered or ranked) or nested words. Then EBSP($\mathcal{S}$) holds with a computable witness function (which is non-elementary, in general).

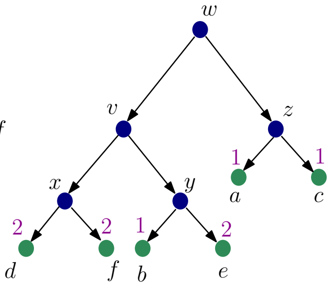
Graphs satisfying EBSP

m-partite cographs

- Hliněný, Nešetřil, et al. introduced in 2012, the class of m-partite cographs.
- An m-partite cograph G is a graph that has an m-partite cotree representation t:



G



t

Label set $= \{1, 2\}$
$f_x = f_z = 0$
$f_y = 1$
$f_v(2, 2) = 1$, else 0
$f_w(1, 1) = 1$, else 0

Important subclasses of m-partite cographs

- Cographs (1-partite cographs): complete graphs, complete k-partite graphs, threshold graphs, Turan graphs, etc.
- Bounded tree-depth graphs
- Bounded shrub-depth graphs

All of the above classes are of active current interest for their excellent algorithmic and logical properties!

m-partite cographs and its subclasses satisfy EBSP

Theorem

Let $\mathcal{S}$ be a hereditary subclass of any of the following graph classes. Then EBSP($\mathcal{S}$) holds with a computable witness function. For classes with bounded parameters as below, there exist elementary witness functions.

1. the class of m-partite cographs
2. any graph class of bounded shrub-depth
3. any graph class of bounded tree-depth
4. the class of cographs

EBSP and well-quasi-orders

A property of binary strings

- Consider the binary strings $u = 00$, $v = 010$ and $w = 111$.
- We say u is a substring of v since u "embeds inside" v. Observe that neither of v or w is a substring of the other.
- For each n, there exists a set of n strings such that no string in the set is a substring of another.
  Eg. $n = 4 : \; 0000, 0101, 1010, 1111$
- However this cannot be done if n becomes infinite!

Theorem (Higman 1952)

For every infinite set $\{w_1, w_2, \ldots\}$ of binary strings, there exist i, j such that w_i is a substring of w_j. In other words, binary strings are well-quasi-ordered (w.q.o.) under the substring relation.

Well-quasi-ordering and EBSP

Definition

A class $\mathcal{S}$ of structures is said to be w.q.o. under embedding if for every infinite set $\{\mathcal{A}_1, \mathcal{A}_2, \ldots\}$ of structures of $\mathcal{S}$, there exist i, j such that $\mathcal{A}_i$ is embeddable in $\mathcal{A}_j$.

Theorem

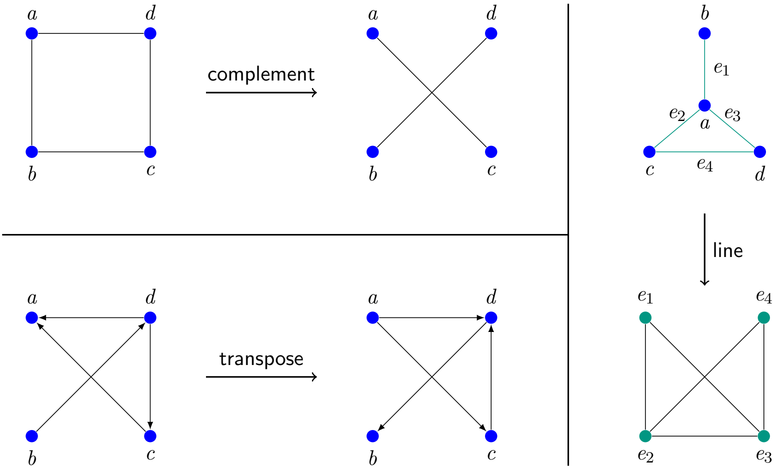
Let $\mathcal{S}$ be w.q.o. under embedding. Then EBSP$(\mathcal{S})$ is true.

Applications: The following classes satisfy EBSP$(\mathcal{S})$:

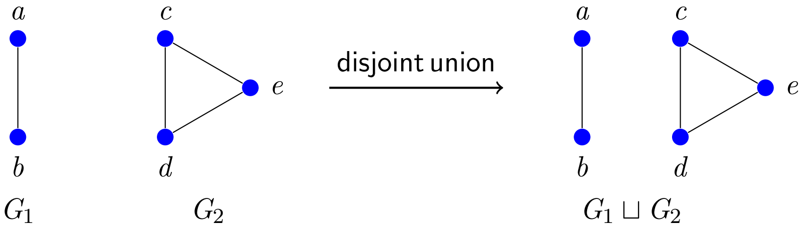
- k-letter graphs for each k (e.g. threshold graphs, unbounded interval graphs)
- k-uniform graphs for each k

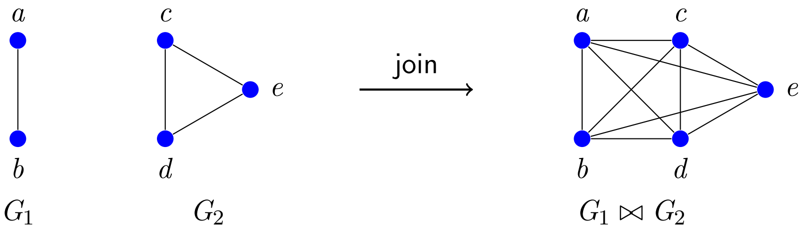
Constructing new classes satisfying EBSP

Unary operations on structures

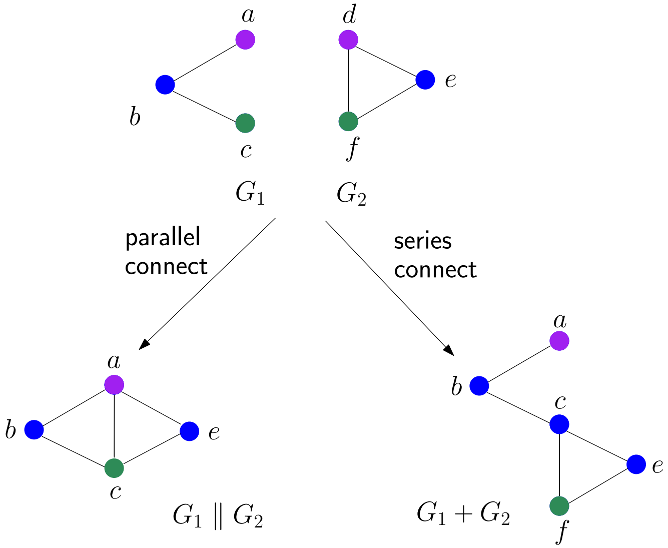


Binary operations on structures

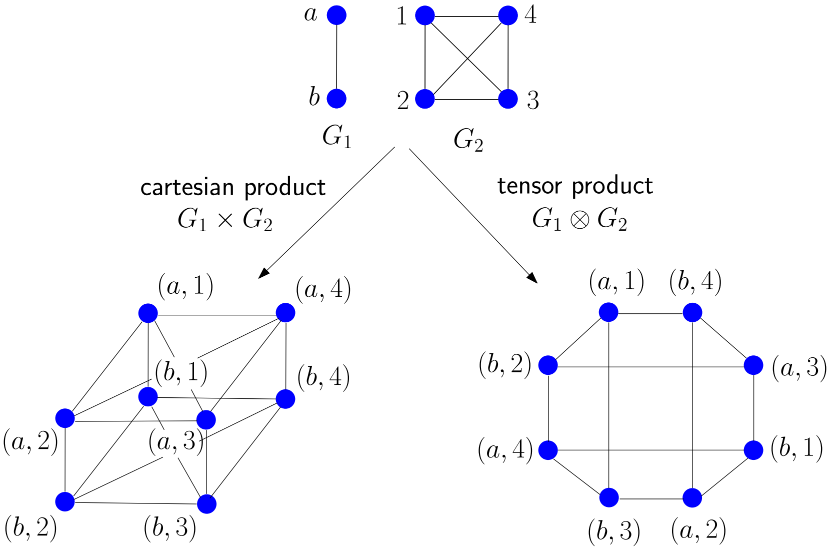




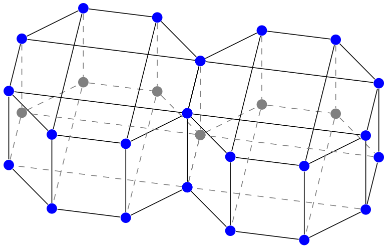
Binary operations on structures

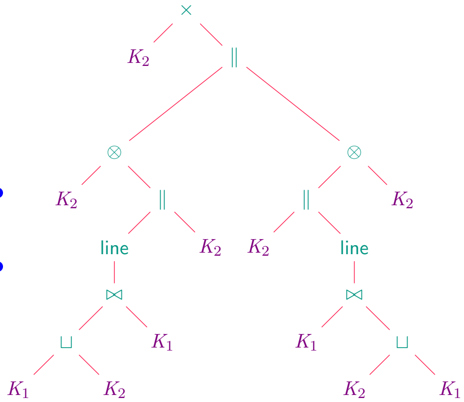


Binary operations on structures



Generating graphs using trees of operations





K_1 = single vertex; K_2 = single edge

Closure of EBSP under unary operations

Theorem

Given a class $\mathcal{S}$, let $\mathcal{Z}$ be any one of the following classes.

1. Complement($\mathcal{S}$)
2. Transpose($\mathcal{S}$)
3. Line($\mathcal{S}$)

Closure of EBSP under unary operations

Theorem

Given a class $\mathcal{S}$, let $\mathcal{Z}$ be any one of the following classes.

1. Complement($\mathcal{S}$)
2. Transpose($\mathcal{S}$)
3. Line($\mathcal{S}$)

Then the following are true:

- EBSP($\mathcal{S}$) $\rightarrow$ EBSP($\mathcal{Z}$)

Closure of EBSP under unary operations

Theorem

Given a class $\mathcal{S}$, let $\mathcal{Z}$ be any one of the following classes.

1. Complement($\mathcal{S}$)
2. Transpose($\mathcal{S}$)
3. Line($\mathcal{S}$)

Then the following are true:

- EBSP($\mathcal{S}$) $\rightarrow$ EBSP($\mathcal{Z}$)
- If EBSP($\mathcal{S}$) holds with a computable/elementary witness function, then so does EBSP($\mathcal{Z}$).

Closure of EBSP under binary operations

Theorem

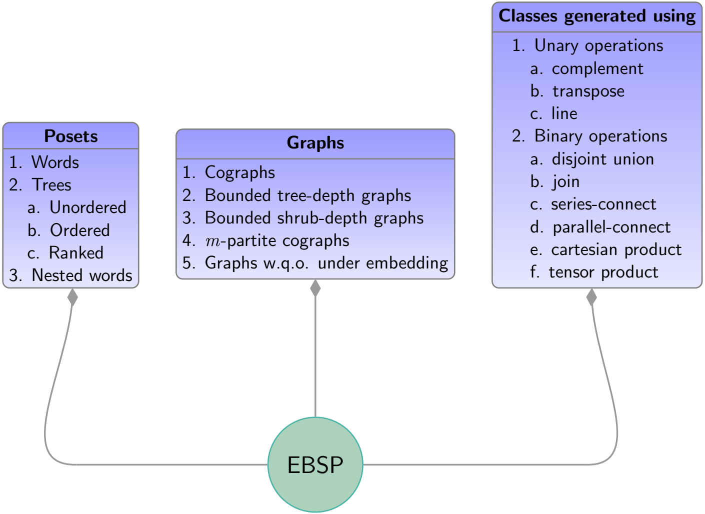
Given classes $\mathcal{S}_1$ and $\mathcal{S}_2$, let $\mathcal{Z}$ be any one of the following classes.

1. Disjoint-union$(\mathcal{S}_1, \mathcal{S}_2)$
2. Join$(\mathcal{S}_1, \mathcal{S}_2)$
3. Series-connect$(\mathcal{S}_1, \mathcal{S}_2)$
4. Parallel-connect$(\mathcal{S}_1, \mathcal{S}_2)$
5. Cartesian-product$(\mathcal{S}_1, \mathcal{S}_2)$
6. Tensor-product$(\mathcal{S}_1, \mathcal{S}_2)$

Then the following are true:

- $\big(\text{EBSP}(\mathcal{S}_1) \wedge \text{EBSP}(\mathcal{S}_2)\big) \rightarrow \text{EBSP}(\mathcal{Z})$
- If the conjuncts in the antecedent hold with computable/ elementary witness functions, then so does the consequent.

An overview of classes satisfying EBSP



Techniques used to prove EBSP for a class

- Key technical features we make use of:
  - Most structures $\mathcal{A}$ seen so far have tree representations $\mathsf{t}_{\mathcal{A}}$.
  - The representations are "good" - the operations used in $\mathsf{t}_{\mathcal{A}}$ satisfy a Feferman-Vaught kind composition property.
  - The index of the "m-similarity" equivalence relation is finite.

Techniques used to prove EBSP for a class

- Key technical features we make use of:
  - Most structures $\mathcal{A}$ seen so far have tree representations $\mathsf{t}_{\mathcal{A}}$.
  - The representations are "good" - the operations used in $\mathsf{t}_{\mathcal{A}}$ satisfy a Feferman-Vaught kind composition property.
  - The index of the "m-similarity" equivalence relation is finite.
- Perform appropriate prunings and graftings iteratively in $\mathsf{t}_{\mathcal{A}}$ to create a subtree that is "rainbow" in the following sense:
  - Height: No two nodes in any path from root to leaf represent structures of the same m-similarity class.
  - Degree: For a node a with children b_i where $1 \leq i \leq n$, let t_i be the subtree rooted at a, from which the subtrees rooted at $b_{i+1}, \ldots, b_n$, are deleted. Then no two t_i's represent structures of the same m-similarity class.

Techniques used to prove EBSP for a class

- Key technical features we make use of:
  - Most structures $\mathcal{A}$ seen so far have tree representations $\mathsf{t}_{\mathcal{A}}$.
  - The representations are "good" - the operations used in $\mathsf{t}_{\mathcal{A}}$ satisfy a Feferman-Vaught kind composition property.
  - The index of the "m-similarity" equivalence relation is finite.
- Perform appropriate prunings and graftings iteratively in $\mathsf{t}_{\mathcal{A}}$ to create a subtree that is "rainbow" in the following sense:
  - Height: No two nodes in any path from root to leaf represent structures of the same m-similarity class.
  - Degree: For a node a with children b_i where $1 \leq i \leq n$, let t_i be the subtree rooted at a, from which the subtrees rooted at $b_{i+1}, \ldots, b_n$, are deleted. Then no two t_i's represent structures of the same m-similarity class.
- The rainbow subtree represents a small m-similar substructure.

Algorithmic meta-theorems for EBSP classes

- The aforementioned prunings and graftings can be implemented in time linear in the size of the tree.
- The rainbow subtree thus obtained in linear time, represents a small uniform kernel for all FO $[m]$ properties of the original structure.
- FO properties of the kernel can be checked in constant time.

Theorem

Let $\mathcal{S}$ be a class of structures admitting good tree representations. Then there exists a linear time f.p.t. algorithm for FO model checking over $\mathcal{S}$, provided input structures are given in the form of their tree representations.

Connection with Nature

Fractals

- Mathematical objects that exhibit self-similarity at all scales.
- Appear widely in Nature.

Fern leaf

Fractals

- Mathematical objects that exhibit self-similarity at all scales.
- Appear widely in Nature.

Conch shell

Fractals

- Mathematical objects that exhibit self-similarity at all scales.
- Appear widely in Nature.

Romanesco cauliflower

A strengthening of EBSP

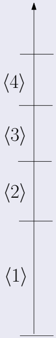
Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if

A strengthening of EBSP

Definition

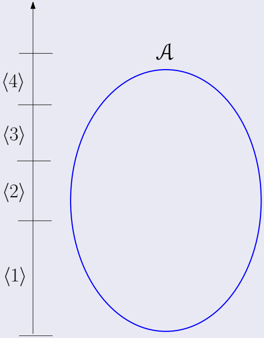
Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if



A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if

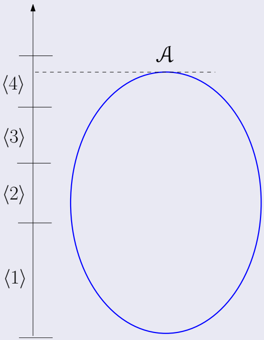


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$

A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if

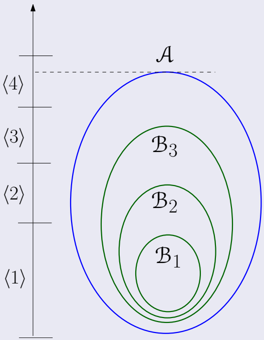


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$

A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if

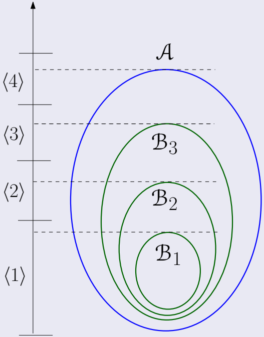


$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$

A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$

If $|\mathcal{A}| \in \langle i \rangle$, then

$\forall j < i$

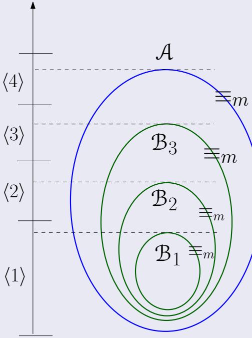
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \;\; \mathcal{B}_j \in \mathcal{S}$

(i) $|\mathcal{B}_j| \in \langle j \rangle$

A strengthening of EBSP

Definition

Given a class $\mathcal{S}$ of finite structures, we say $\text{EBSP}^{\#}(\mathcal{S})$ holds if



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$

If $|\mathcal{A}| \in \langle i \rangle$, then

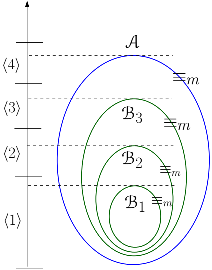
$\forall j < i$

$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$

(i) $|\mathcal{B}_j| \in \langle j \rangle$

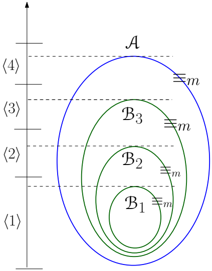
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

EBSP$^{\#}$ – a fractal-like property



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$
(i) $|\mathcal{B}_j| \in \langle j \rangle$
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

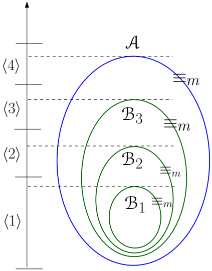
EBSP$^{\#}$ – a fractal-like property



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i\rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$
(i) $|\mathcal{B}_j| \in \langle j\rangle$
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

- EBSP$^{\#}$ indeed asserts logical self-similarity at all scales.

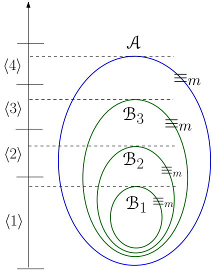
EBSP$^{\#}$ – a fractal-like property



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$
(i) $|\mathcal{B}_j| \in \langle j \rangle$
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

- EBSP$^{\#}$ indeed asserts logical self-similarity at all scales.
- All the classes seen so far can be shown to satisfy EBSP$^{\#}$.

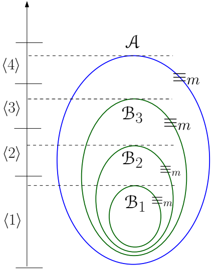
EBSP$^{\#}$ – a fractal-like property



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$
(i) $|\mathcal{B}_j| \in \langle j \rangle$
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

- EBSP$^{\#}$ indeed asserts logical self-similarity at all scales.
- All the classes seen so far can be shown to satisfy EBSP$^{\#}$.
- Whereby all these classes can be regarded as

EBSP$^{\#}$ – a fractal-like property



$\forall \mathcal{A} \in \mathcal{S} \quad \forall m \in \mathbb{N}$
If $|\mathcal{A}| \in \langle i \rangle$, then
$\forall j < i$
$\exists \mathcal{B}_j \subseteq \mathcal{A}, \quad \mathcal{B}_j \in \mathcal{S}$
(i) $|\mathcal{B}_j| \in \langle j \rangle$
(ii) $\mathcal{B}_j$ is m-similar to $\mathcal{A}$

- EBSP$^{\#}$ indeed asserts logical self-similarity at all scales.
- All the classes seen so far can be shown to satisfy EBSP$^{\#}$.
- Whereby all these classes can be regarded as logical fractals!

Conclusion

Summary of the talk

A. Notions:

- The Downward Löwenheim-Skolem Property: DLSP
- The Equivalent Bounded Substructure Property: EBSP

B. Results:

- Classes of finite structures satisfying EBSP
- Closure properties of EBSP
- Techniques and f.p.t. algorithms
- Connection with Nature

A challenging open problem

Is there a structural characterization of EBSP/logical fractals?

Dhanyavād!

References

- A. Sankaran. A finitary analogue of the downward Löwenheim-Skolem property. *Proceedings of CSL 2017, Stockholm, Sweden, August 20-24, 2017, to appear*.
- A. Sankaran, B. Adsul, and S. Chakraborty. A generalization of the Łoś-Tarski preservation theorem. *Annals of Pure and Applied Logic*, 167(3):189 - 210, 2016.
- A. Sankaran, B. Adsul, and S. Chakraborty. A generalization of the Łoś-Tarski preservation theorem over classes of finite structures. In *Proceedings of MFCS 2014, Budapest, Hungary, August 25-29, 2014, Part I*, pages 474 - 485, 2014.