The Skolem problem: variants, applications and results — A survey talk

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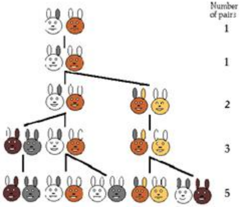
The Fibonacci Sequence



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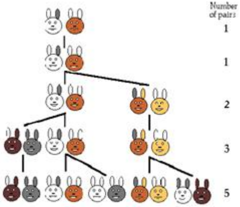

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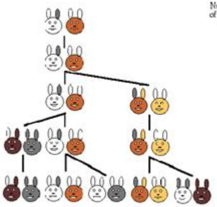

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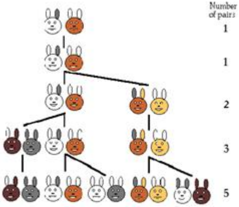

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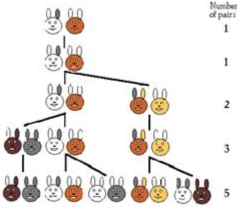

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- The Question: Can they ever die out?

Linear Recurrence Sequences (LRS)

Definition

A sequence $\langle u_0, u_1, \ldots \rangle$ of numbers is called an LRS if there exists $k \in \mathbb{N}$ and constants $a_0, \ldots, a_{k-1}$ s.t., for all $n \geq k$,

$$u_n = a_{k-1} u_{n-1} + \ldots + a_1 u_{n-k+1} + a_0 u_{n-k}$$

- k is called the order/depth of the sequence.
- The first k elements $u_0, \ldots, u_{k-1}$ are called initial conditions and they determine the whole sequence.
- We can define the sequences and constants to be over integers or rationals or reals.

The Skolem Problem

Figure: Thoralf Skolem

The Skolem Problem (also called the Skolem-Pisot Problem)

Given a linear recurrence sequence (with initial conditions) over integers, does it have a zero?

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Given a linear recurrence sequence (with initial conditions) over integers, does it have a zero?

i.e., does $\exists n$ such that $u_n = 0$?

i.e., do the rabbits ever die out?

The Skolem Problem

Skolem Problem: Does $\exists n$ such that $u_n = 0$?

Surprisingly, this problem has been open for 80 years!

Well, in 1934 decidability wasn't as relevant...

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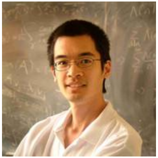

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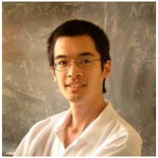

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"...a mathematical embarrassment..." – Richard Lipton, Chap. 42, The P=NP question and Gödel's lost letter, Springer, 2010.

Outline

- Alternative formulations and variants
- Applications
  1. Program Termination
  2. Probabilistic verification
- Results
  1. Classical results on Skolem
  2. Relation between the problems
  3. Results on Program termination
  4. Recentmost Results

Equivalent formulations of the Skolem Problem

Linear recurrence sequence form

Given an LRS $\langle u_1, u_2, \ldots \rangle$ (with initial conditions), does $\exists n$ s.t., $u_n = 0$?

Matrix Form

Given a $k \times k$ matrix M, does $\exists n$ s.t., $M^n(1, k) = 0$?

Dot Product Form

Given a $k \times k$ matrix M, k-dim vectors $\vec{v}, \vec{w}$, does $\exists n$ s.t., $\vec{v} \cdot M^n \cdot \vec{w}^T = 0$?

Equivalent formulations

(1) $\implies$ (2) Let $u_n = a_{k-1}u_{n-1} + \ldots + a_0 u_{n-k}$, $a' = (a_{k-1} \ldots a_1)$.

- Let

$$M_1 = \begin{pmatrix} \vec{a'} & Id_{k-1} \\ a_0 & 0 \end{pmatrix}$$

  Then $\forall n \geq 0,\ u_n = \vec{v} \cdot M_1^n \cdot \vec{w}^{\,T}$, where $\vec{v} = \vec{u}$, $\vec{w} = (0 \ldots 0\ 1)$.
- Define

$$M = \begin{pmatrix} 0 & \vec{v} \cdot M_1 \\ \vec{0}^T & M_1 \end{pmatrix} \qquad M^n = \begin{pmatrix} 0 & \vec{v} \cdot M_1^n \\ \vec{0}^T & M_1^n \end{pmatrix}$$

- Then, $M^n(1, k+1) = (1\ 0) \cdot (M^n) \cdot (0\ \vec{w})^T = u_n$.

(3) $\implies$ (1) follows from taking the characteristic polynomial and using Cayley Hamilton Theorem.

Variants

Skolem Problem

- Given an LRS $\langle u_1, u_2, \ldots \rangle$, does $\exists n$ s.t., $u_n = 0$?

Positivity Problem

- Given an LRS $\langle u_1, u_2, \ldots \rangle$, $\forall n$, is $u_n \geq 0$?
- Ultimate Positivity: $\forall n, n \geq T$, is $u_n \geq 0$?

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Applications of these and other variants!

- Software verification
  - Termination of linear programs
- Probabilistic model checking
  - Reachability in Markov chains
- Theoretical Biology
  - Analysis of L-systems, Population dynamics
- Economics
  - Stability of supply-demand equilibria in cyclical markets
- Quantum Computing
  - Threshold problems for quantum automata
- Dynamical systems
  - Reachability and invariance problems
- Combinatorics
- Term rewriting
- ...

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Program Termination

Basic undecidability result – Turing 1936

Termination of a generic program with a loop:

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But now, let us consider a much simpler case:

A simple linear program

$$\vec{x} := \vec{b};\ \ \textbf{while}\ (\vec{c}^{\,T}\vec{x} > \vec{0})\ \ \{\vec{x} := A\vec{x}\}$$

Linear Programs

An initialized (homogenous) linear program

$\vec{x} := \vec{b};$ **while** $(\vec{c}^T\vec{x} > \vec{0})$ $\{\vec{x} := A\vec{x}\}$

Termination problem for simple linear programs

Does an instance of the above program i.e., $\langle \vec{b}; \vec{c}; A \rangle$, terminate?

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Theorem [Rohit Singh, Supratik Chakraborty]

Consider the following single input initialized linear loop program:

$$\vec{x} := \vec{b}; \ \textbf{while}\ (B\vec{x} > \vec{e}) \ \{\vec{x} := A\vec{x} + \vec{d}\}$$

The termination problem for this program is equivalent to the positivity problem.

Reachability in Markov chains

Consider a Markov chain M over states $s_1, \ldots, s_t$.

Question

Starting from a given initial probability distribution $\vec{v}$, is it the case that eventually the probability of staying in state s_t will stay within $[0, 1/2]$?

For e.g., the nodes above could be protein concentrations, and the Markov chain a model of biochemical reactions and we want to check for high conc.

Reachability in Markov chains

Example: Consider $\vec{v} = (1/4, 1/4, 1/2)$ and

$$M = \begin{pmatrix} 0.6 & 0.1 & 0.3 \\ 0.3 & 0.6 & 0.1 \\ 0.1 & 0.3 & 0.6 \end{pmatrix}$$

- Then, does $\exists T$ s.t., for all $t > T, \vec{v} \cdot M^t \cdot (1\ 0\ 0) > 1/3$?
- Then, does $\exists t$ s.t., $\vec{v} \cdot M^t \cdot (1\ 0\ -1) = 0$?

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Theorem

The Skolem over rationals can be reduced to (two!) Stochastic version(s):

- Given stochastic vector $\vec{v}$, vector w, row-stochastic matrix M, does $\exists t$ s.t., $\vec{v} \cdot M^t \vec{w} = 1/2$
- Given stochastic vectors $\vec{v}, \vec{w}$, row-stochastic matrix M, rational r, does $\exists t$ s.t., $\vec{v} \cdot M^t \vec{w} = r$

Results

1. Classical results on Skolem
    - Skolem-Mahler-Lech Theorem
    - Decidability of Skolem/Positivity for 2,3,4...
2. Relation between the problems
3. Results on Program termination- Tiwari, Braverman, Supratik et al.
4. Recentmost results - Ouaknine, Worrell, et al.
    - Orbit problem - extension of Kannan/Lipton
    - Positivity problem
    - Other/probabilistic results and reductions

Classical Results on Skolem

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- In other words, the hardness of the result is in characterizing the finite set.
- All known proofs of the above result use p-adic integers.

Lower order results

Skolem's problem

- Order 1: Trivial (why?).
- Order 2: Folklore!
- Order 3,4: Proved by Vereshchagin in 1985 using results on linear logarithms by Baker and van der Poorten.
    - This theory fetched Baker the Field's medal in 1970!

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(Ultimate) Positivity problem

- Order 2: Burke and Webb (1981) – Ultimate
- Order 2: Halava, Harju and Hirvensalo (2006) – integer LRS
- Order 3: Laohakosol and Tangsupphathawat (2009)

Hardness results

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- The (complement of) Skolem problem reduces to Positivity
    - LRS are closed under pointwise product and sum
    - given LRS u_n, $u_n \neq 0$ iff $u_n^2 - 1 \geq 0$.
- Thus, Positivity is coNP hard.
- Ultimate Positivity is also coNP hard.

The Orbit problem

Orbit Problem

- Given a $k \times k$ matrix M, k-dim vectors $\vec{x}$ and $\vec{y}$, does $\exists n$ s.t., $\vec{x} \cdot M^n = \vec{y}$?
- Higher Order Orbit Problem: Given $k \times k$ matrix M, k-dim vector $\vec{x}$, a subspace V of dim $\leq k$, does $\exists n$ s.t., $\vec{x} \cdot M^n \in V$?

- Skolem problem (does $\exists n$ s.t., $\vec{v} \cdot M^n \cdot \vec{w}^T = 0$?) is special case of the higher order Orbit Problem
- Thus, Higher order Orbit Problem is also NP hard.

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Kannan, Lipton – STOC'80, JACM'86

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Chonev,Ouaknine, Worrell– STOC'12

- High dim Orbit Problem for dim 1 is in P
- High dim Orbit Problem for dim 2 or 3 is in NP^{RP}

Termination of Linear Programs

Non-homogenous to Homogenous

$$\vec{x} := \vec{b};\ \ \textbf{while}\ (B\vec{x} > \vec{e})\ \ \{\vec{x} := A\vec{x} + \vec{d}\}$$

By adding a new scalar variable z,

$$\vec{x} := \vec{b}, z = 1;\ \ \textbf{while}\ (B\vec{x} - \vec{e}z > 0)\ \ \{\vec{x} := A\vec{x} + \vec{d}z; z = z\}$$

Thus, we only have to consider:

$$\vec{x} := \vec{b}\ \ \textbf{while}\ (B\vec{x} > 0)\ \ \{\vec{x} := A\vec{x}\}$$

- Tiwari CAV'04 : **while** $(B\vec{x} > 0)\ \ \{\vec{x} := A\vec{x}\}$ termination is decidable over reals
- Braverman CAV'06: The above problem is decidable over integers
- Singh, Supratik: Reduction to Positivity, decidability for subclass

Recentmost Results

Ouaknine, Worrell – Announced on webpage

- positivity for LRS of order 5 or less is decidable with complexity $coNP^{PP^{PP^{PP}}}$.
- ultimate positivity for LRS of order 5 or less is decidable in P.
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"All prior work on Positivity problems that we are aware of has been confined to the use of linear algebra and elementary algebraic number theoretic techniques. By contrast, we are deploying an eclectic arsenal of deep and sophisticated tools from analytic and algebraic number theory, Diophantine geometry, . . ."

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Some high-level intuition:

1. check if u_n ultimately positive by looking at its "exponential polynomial soln" in P.
2. if u_n is ult. pos. with $ord < 5$, we can compute N (of at most exp magnitude) s.t., u_n is positive after N.

Acknowledgments

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