Proving Properties of Concurrent Data Structures

Papers from LICS'13 and CONCUR'13

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CMI

28 July 2013

Outline

1. Quantitative Reasoning for Proving Lock-Freedom - Jan Hoffman, Michael Marmar, Zhong Shao: In the proceedings of LICS 2013

2. Aspect-Oriented Linearizability Proofs: Thomas A. Henzinger, Ali Sezgin, and Viktor Vafeiadis: In the proceedings of CONCUR 2013

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Summary of the contributions:

Reduces proving *lock-freedom* to modular thread local termination of concurrent programs in which each thread executes a finite number of data-structure operations.

Introduces a *compensation based* quantitative reasoning technique for proving lock-freedom.

Formalises the technique by extending *Concurrent Separation Logic (CSL)* for total correctness.

Demonstrates the lock-free property exhibited by data structures including Treiber's non-blocking stack, Michael and Scott's lock-free queue, Hendler et al.'s lock-free stack with elimination back off and Michale's lock-free hazard pointer stack.

Lock Freedom

Consider a shared memory data structure which provides the users with finitely many operations to access/modify the contents of the data-structure.

Assume that at a given time there is a fixed but arbitrary number of threads that are repeatedly accessing the data-structure via the operations it provides.

Choose a point in the execution in which one or more operations have started.

Definition

Then lock-free *implementation of the data-structure guarantees that* some *thread will complete an operation in a finite number of steps.*

Lock Freedom and Termination

Definition

Let D *be a shared-memory data structure with* k*-operations denoted by* $\pi_1, \pi_2, \ldots, \pi_k$.

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$$P = S_1 \,||\, \ldots \,||\, S_m$$

where each S_i is a sequential program executing finitely many (say n_i) D-operations.

$$S_i = op_1; op_2; \ldots; op_{n_i} \text{ where } \forall j \in [1, \ldots, n_i], op_j \in \{\pi_1, \ldots \pi_k\}$$

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Theorem

The data-structure D with operations $\pi_1, \ldots, \pi_n$ is lock free iff every such program P terminates.

Lock-free data structure: An example

Example

Let A be a heap location of type `Int`, shared between a a number of producer and consumer threads.

A producer checks if A is 0, and if so, it updates A with a newly produced non-zero value and terminates.

A consumer checks if A contains a non-zero value, and if so, consumes the value, sets the value of A to 0 and loops to check if A contains a new value to consume. If A contains 0 then it terminates.

We want to prove that if a consumer does not terminate then it is busy performing some useful work, i.e, consuming the data-produced by the producer.

Producer-Consumer Code

```
producer(int y):
    atomic ⟨
        if ([A] == 0):
            [A] = y;
        else :
            skip; ⟩
```

```
consumer():
    Int x = 1;

    while (x ≠ 0):
        atomic ⟨
            b = [A];
            if (b ≠ 0):
                x = b;
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Lock Freedom: Observation

Informal reasoning about lock-freedom

In an operation of a lock-free data-structure, the failure of a thread to make progress is always caused by successful progress in an operation executed by another thread.

A thread which fails to make progress, typically retries the operation.

In concurrent execution of finitely many threads, each performing finitely many operations of a lock-free data structure, one can precompute the upper bound on the number of retries that each thread can perform.

Example

If m_c consumer threads and m_p producer threads were running concurrently, then the total number of loop iterations across all the consumer threads is at most $m_c + m_p$.

Introducing Quantitative reasoning

Definition (Affine Resource)

An affine resource is one which once consumed cannot be regenerated.

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Quantitative Reasoning

Each thread begins with a finite number of tokens which are affine resources.

Each time a thread wants to *try* performing the operation, it pays the price of one token which gets consumed.

When a thread's operation succeeds, it doesn't need to retry. Hence it can *compensate* for the failure of other threads by *transferring* the remaining tokens to the other threads which failed to make progress.

When a thread's operation fails, it is compensated by the thread which makes progress and can thus pay for the subsequent retry.

Introducing Quantitative reasoning

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Quantitative Reasoning

The total number of tokens the system begins with provides the upper bound on the number of retries.

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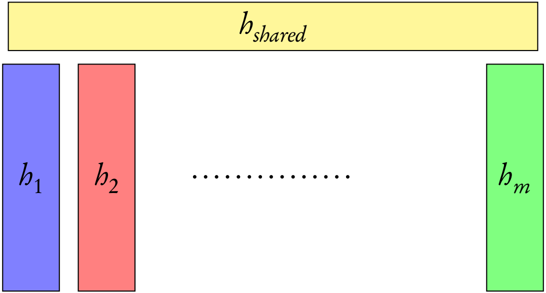
Concurrent Separation Logic (CSL): A quick and dirty introduction

In a concurrent program of m threads, the memory is partitioned into disjoint portions $h_1, h_2, \ldots h_m$ and h_s where

- $\forall\, i \in [1, \ldots, m]$, h_i is the set of all memory locations accessible only to thread i called the private heap of i.
- h_{shared} is the remaining set of memory locations shared between the threads called the *shared heap*.



Concurrent Separation Logic (CSL): A quick and dirty introduction

Heaps are characterised using separation logic assertions.

$$P, Q ::= \mathit{true} \mid \mathit{emp} \mid [x] \mapsto y \mid \neg P \mid P \wedge Q \mid P \vee Q \mid P * Q \mid \exists z.P \mid \forall z.P$$

For any heap h, $h \vDash [x] \mapsto y$ iff h is a single memory cell x which stores the value y.

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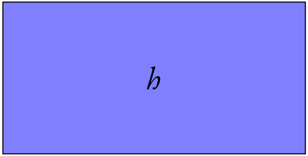
Suppose P and Q are assertions, we say that a heap $h \vDash P * Q$ iff



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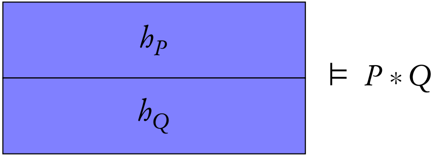

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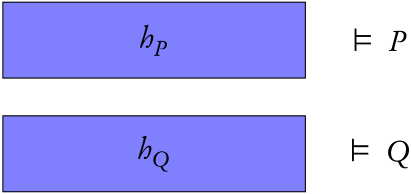

Concurrent Separation Logic (CSL): A quick and dirty introduction

Let I, P, Q denote separation logic assertions describing the heaps.

Concurrent Separation Logic judgement

The judgement $I \vdash [P]\ C\ [Q]$ is to be understood as follows:

> A thread executing program C beginning with a private heap that satisfies P executes *safely* and *terminates* resulting in a private heap of the thread which satisfies Q.
>
> Throughout the execution of C (except inside the atomic sections), the shared heap satisfies I.

Concurrent Separation Logic (CSL): A quick and dirty introduction

Rule for parallel composition: PAR

$$\frac{I \vdash [P_1]\, C_1\, [Q_1] \quad \ldots \quad I \vdash [P_m]\, C_m\, [Q_m]}{I \vdash [P_1 * \cdots * P_m]\, C_1 \,||\, \cdots \,||\, C_m\, [Q_1 * \cdots * Q_m]}$$

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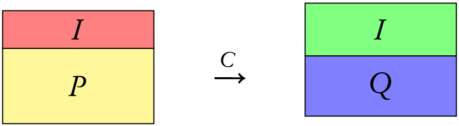

Rule for Atomic sections: ATOM

$$\vdash [P * I]\, C\, [Q * I]$$

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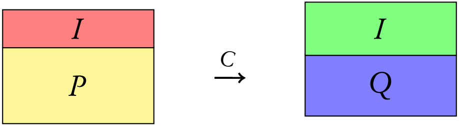

Rule for Atomic sections: ATOM

$$\frac{\vdash [P * I]\, C\, [Q * I]}{I \vdash [P]\langle C\rangle[Q]}$$

Back to the Paper: Quantitative CSL

Let $\Diamond$ be a predicate such that for any heap h, $h \vDash \Diamond$ iff the heap h has at least one affine token.

We write $\Diamond^k$ as a shorthand for $\underbrace{\Diamond * \cdots * \Diamond}_{k \text{ times}}$.

Quantitative CSL

Rule for `while` loop in CSL:

$$\frac{I \vdash [P \wedge B]\ C\ [P]}{I \vdash [P]\ \texttt{while}(B)\ \texttt{do}\ \ C\ [P \wedge \neg Cond]}$$

Quantitative CSL

Rule for `while` loop in Quantitative CSL:

$$\frac{P \wedge B \implies P' * \Diamond \quad I \vdash [P']\, C\, [P]}{I \vdash [P]\ \texttt{while}(B)\ \texttt{do}\ C\ [P \wedge \neg B]}$$

Using Quantitative CSL to prove lock freedom of Producer-Consumer

Example

Setting $I := A \mapsto 0 \vee ((\exists u : u \neq 0 \wedge A \mapsto u) * \Diamond)$
Loop invariant $P := x = 0 \vee \Diamond$,
loop condition $B := x \neq 0$
and the use of ATOM rule, we can show that

$$I \vdash [\Diamond]\texttt{consumer()}[emp]$$

and

$$I \vdash [\Diamond]\texttt{producer()}[emp]$$

Using Quantitative CSL to prove lock freedom of Producer-Consumer

Example

If S_i is a sequential program invoking exactly n_i calls from $\{\texttt{producer()}, \texttt{consumer()}\}$ then by induction we can prove that

$$I \vdash [\Diamond^{n_i}]\, S_i\, [emp]$$

.

If P is a concurrent program $S_1 \,||\, S_2 \,||\, \cdots \,||\, S_m$ then by PAR rule we have

$$I \vdash [\Diamond^{n_{tot}}]\, P\, [emp]$$

where $n_{tot} = \Sigma_{i=0}^{m} n_i$.

This proves the termination of P.

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Contributions of this paper

Reduces the task of verifying linearizability of a queue implementation to establishing four basic properties each of which can be independently verified.

Demonstrates the linearizability of Herlihy-Wing queue using the proposed technique.

Uses RGSep, a combination of Rely-Guarantee Logic and Separation Logic to automate the verification of three of these four properties.

Linearizability

Suppose Q is a concurrent queue over the domain $Val = \mathbb{N} \cup \{\texttt{NULL}\}$ that supports two methods

- $\textsf{enq}(x : \mathbb{N})$ that enqueues the value x into the queue. Returns void.

  We denote an instance of this method call by $\langle \textsf{enq},\ x \rangle$.

  Each $\langle \textsf{enq},\ x \rangle$ method instance has an invocation event denoted by $\langle \textsf{enq},\ x \rangle_i$ and a response event denoted by $\langle \textsf{enq},\ x \rangle_r$.

- $\textsf{deq}(\texttt{void})$ which returns some value y from Val.

  We denote an instance of this method call by $\langle \textsf{deq},\ y \rangle$.

  Each $\langle \textsf{deq},\ y \rangle$ method instance has an invocation event denoted by $\langle \textsf{deq},\ y \rangle_i$ and a response event denoted by $\langle \textsf{deq},\ y \rangle_r$.

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Definition (History)

A history c*, is a sequence of invocation and response events where every response event has a corresponding invocation event that appears before it in the sequence.*

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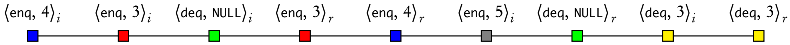

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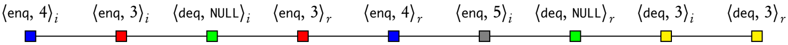

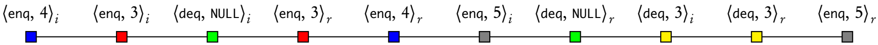

Note

In a history c not every invocation events needs to have a corresponding response event. Such histories are called *incomplete histories*. Eg. $\langle \text{enq}, 5\rangle_i$

An incomplete history c can be *completed* by appending the response events for the unmatched invocation events to obtain it's completion $\hat{c}$.

There could be several completions of an incomplete history.

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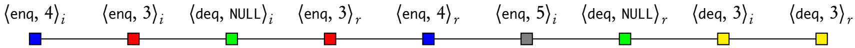


Definition (Happened Before)

Let c *be a history and* $<_c$ *the total order on the set of events in* c.

We say that the method call m happened-before *a method call* m' *in* c*, denoted by* $m \xrightarrow{\text{hb}}_c m'$ *iff* $m_r <_c m'_i$.

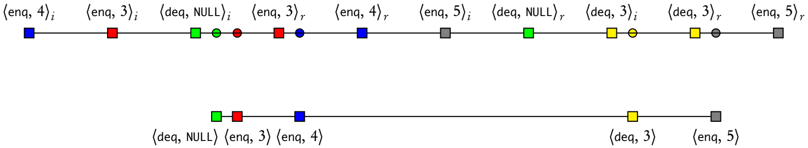
Eg: $\langle \text{enq}, 4\rangle \xrightarrow{\text{hb}}_c \langle \text{deq}, 3\rangle$.

Linearizability

Definition (Linearlizability)

A history c is said to be linearizable *iff there exists some completion $\hat{c}$ of c in which*

> *For every method m there is a* linearization point *at some instant between m_i and m_r.*
>
> *All methods appear to occur instantly at their linearization point, behaving as specified by the sequential specification.*



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Proving Linearizability of a concurrent queue implementation

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However, this technique doesn't lend itself to proving linearizability of several concurrent queue implementations. Eg: Herlihy-Wing queue.

Proving Linearizability

```
int q.back = 0
Val q.items[] = {NULL, NULL, ...}

void enq(int x):
    int i;
    atomic ⟨
        i = q.back;
        q.back++; ⟩// E_1

    atomic ⟨
        q.items[i] = x ⟩// E_2
```

```
Val deq():
    int i, range;
    Val x;

    while (true):
        atomic ⟨
            range = q.back − 1; ⟩// D_1

        for i from 0 to range:
            atomic ⟨
                x = q.items[i]
                q.items[i] = NULL; ⟩// D_2

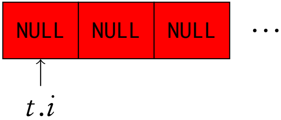
            if (x ≠ NULL) return x;
```

Proving Linearizability

Let t, u, v, w be four concurrent threads. Let $\circ$ denote context switch.

Consider the execution fragment:

$$c = (t : E_1)$$



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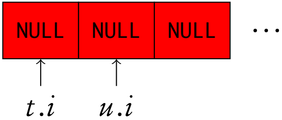
$$c = (t : E_1) \circ (u : E_1)$$



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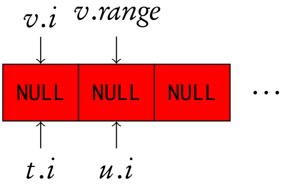

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Let t, u, v, w be four concurrent threads. Let $\circ$ denote context switch.
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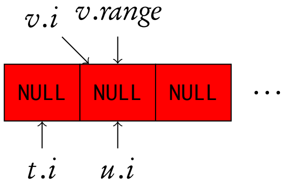

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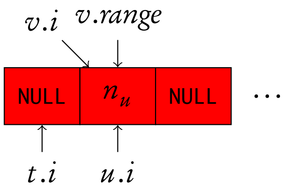

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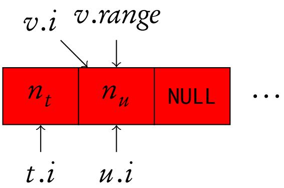

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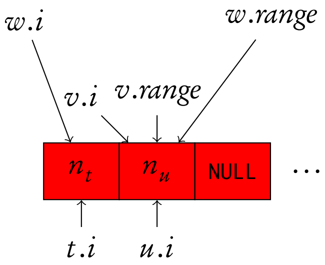

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At the end of this execution fragment,

- t has enqueued an item in $q.items[0]$.
- u has enqueued an item in $q.items[1]$.
- v is ready to dequeue the value enqueued by u.
- w is ready to dequeue the value enqueued by t.

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Suppose we choose E_1 in **enq** to be the linearization point for t then the following extension of c is not linearizable via these linearization point.

$$(v : D_2, return) \circ (z : D_1, D_2, return)$$

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$$(v : D_2, return) \circ (z : D_1, D_2, return)$$

$t : \langle \textsf{enq},\ n_t \rangle$ takes effect before $u : \langle \textsf{enq},\ n_u \rangle$
$v : \langle \textsf{deq},\ n_u \rangle$ takes effect before $z : \langle \textsf{deq},\ n_t \rangle$.

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$$c = (t : E_1) \circ (u : E_1) \circ (v : D_1, D_2) \circ (u : E_2) \circ (t : E_2) \circ (w : D_1)$$

Similarly if we choose E_2 in **enq** to be the linearization point for t then we have the following extension of c which is not linearizable via this linearization point.

$$(w : D_2, return) \circ (z : D_1, D_2, D_2, return)$$

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Similarly if we choose E_2 in **enq** to be the linearization point for t then we have the following extension of c which is not linearizable via this linearization point.

$$(w : D_2, return) \circ (z : D_1, D_2, D_2, return)$$

$u : \langle \textbf{enq}, n_u \rangle$ takes effect before $t : \langle \textbf{enq}, n_t \rangle$
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Aspect oriented Linearizability Proof

Intuitively, a *correct* concurrent history of a queue implementation should not have any of the four violations.

(**VFresh**): A dequeue event returning a value not inserted by any enqueue event.

(**VRepeat**): Two dequeue events returning the value inserted by the same enqueue event.

(**Vord**): Two ordered dequeue events returning values inserted by ordered enqueue events in the inverse order.

(**VWit**): A dequeue event returning `NULL` even though the queue is never logically empty during the execution of the dequeue event.

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Theorem

A set of histories C of a concurrent queue is linearizable iff for every $c \in C$ there exists a completion $\hat{c}$ that has none of the **VFresh**, **VRepeat**, **Vord** , **VWit** *violations.*

Few remarks

Consider the following history

$$\hat{c} = \langle \mathsf{enq},\ 1\rangle_i \cdot \langle \mathsf{enq},\ 1\rangle_r \cdot \langle \mathsf{enq},\ 2\rangle_i \cdot \langle \mathsf{enq},\ 2\rangle_r \cdot \langle \mathsf{deq},\ 2\rangle_i \cdot \langle \mathsf{deq},\ 2\rangle_r \cdot \langle \mathsf{enq},\ 3\rangle_i \cdot \langle \mathsf{enq},\ 3\rangle_r$$

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Since it is a sequential history, we can rewrite it as

$$\hat{c} = \langle \mathsf{enq},\ 1\rangle \cdot \langle \mathsf{enq},\ 2\rangle \cdot \langle \mathsf{deq},\ 2\rangle \cdot \langle \mathsf{enq},\ 3\rangle$$

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One may verify that it is a complete history and has none of the four violations.

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Observation

For any complete history $\hat{c} \in C$, for any finite k, there exists values $v_1, \ldots, v_k \in \mathbb{N} \cup \{\text{NULL}\}$ such that the extension $\hat{c} \cdot \langle \text{deq},\ v_1\rangle_i \cdot \langle \text{deq},\ v_1\rangle_r \cdot \cdots \cdot \langle \text{deq},\ v_k\rangle_i \cdot \langle \text{deq},\ v_k\rangle_r \in C$.

Few remarks

Consider the following sequential history of a queue.

$$\hat{c}_{ext} = \langle \mathsf{enq},\ 1 \rangle \cdot \langle \mathsf{enq},\ 2 \rangle \cdot \langle \mathsf{deq},\ 2 \rangle \cdot \langle \mathsf{enq},\ 3 \rangle \cdot \langle \mathsf{deq},\ v_1 \rangle \cdot \langle \mathsf{deq},\ v_2 \rangle$$

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If $v_1 \neq 3$ then $\hat{c}_{ext}$ is going to violate one of the four violations.

- If $v_1 \notin \{1, 2, 3, \texttt{NULL}\}$, $\hat{c}_{ext}$ violates **VFresh**.
- If $v_1 = 2$, $\hat{c}_{ext}$ violates **VRepeat**.
- If $v_1 = 1$, $\hat{c}_{ext}$ violates **VOrd**.
- If $v_1 = \texttt{NULL}$, $\hat{c}_{ext}$ violates **VWit**.

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We cannot assign any value to v_2 without violating one of the four properties.

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Since the complete history $\hat{c}_{ext} \in C$ and it has at least one of these violations, by the theorem, C is not linearizable.

Thank You!