A Generalization of the Łoś-Tarski Preservation Theorem

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Introduction

- Preservation theorems have been one of the earliest areas of study in classical model theory.
- A preservation theorem characterizes (definable) classes of structures closed under a given model theoretic operation.
- Preservation under substructures, extensions, unions of chains, homomorphisms, etc.
- Most preservation theorems fail in the finite.
- Some preservation results recovered over special classes of finite structures, like those with bounded degree, bounded tree-width etc. (Dawar et al.)
- Homomorphism preservation theorem is true in the finite (Rossman).

Some assumptions and notations for the talk

- Assumptions:
  - First Order (FO) logic.
  - Relational vocabularies (i.e. only predicates).
  - Arbitrary structures typically, unless stated otherwise explicitly.
- Notations:
  - $\Sigma_1 = \exists^*(\ldots), \Pi_1 = \forall^*(\ldots)$
    $\Sigma_2 = \exists^*\forall^*(\ldots), \Pi_2 = \forall^*\exists^*(\ldots)$
  - $M_1 \subseteq M_2$ means M_1 is a substructure of M_2. For graphs, $\subseteq$ means *induced subgraph*.
  - U_M = universe of M.

Preservation under Substructures

Definition 1 (Pres. under subst.)

A sentence ϕ is said to be **preserved under substructures**, *denoted $\phi \in PS$, if $\big((M \models \phi) \wedge (N \subseteq M)\big) \rightarrow N \models \phi$.*

- E.g.: Consider $\phi = \forall x \forall y E(x, y)$ which describes the class of all cliques.
- Any induced subgraph of a clique is also a clique. Then $\phi \in PS$.
- In general, every Π_1 sentence (i.e. $\forall^*$ sentence) is in PS.

Theorem 1 (Łoś-Tarski, 1960s)

A FO sentence in PS is equivalent to a Π_1 sentence.

Preservation under substructures modulo Bounded Cores

Definition 2 (Pres. under subst. modulo bounded cores)

A sentence ϕ is said to be **preserved under substructures modulo a core of size** k*, denoted $\phi \in PSC(k)$, if for each model M of ϕ, there is a subset C of U_M, of size $\leq k$, s.t. $\big((N \subseteq M) \wedge (C \subseteq U_N)\big) \rightarrow N \models \phi$.*

- The set C is called a **core of** M **w.r.t.** ϕ. If ϕ is clear from context, we will call C as a **core of** M.
- For every $\phi \in PS$, for each model M of ϕ, the empty subset is a core of M. Then $PS \subseteq PSC(0)$. Easy to see that $PSC(0) \subseteq PS$. Then $PS = PSC(0)$.

Example

- E.g.: Consider $\phi = \exists x \forall y E(x, y)$.

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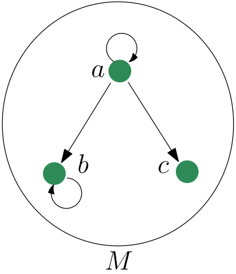

$M \models \phi$

Example

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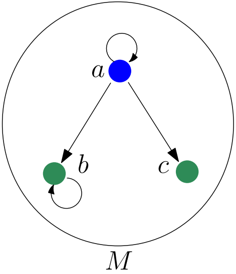

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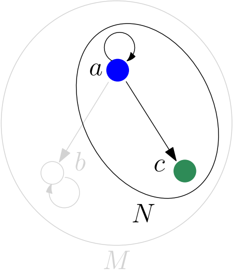

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Example

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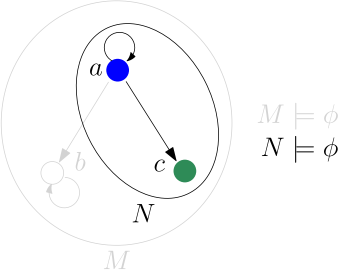
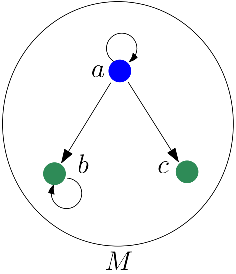


- Any witness for x is a core. Thus $\phi \in PSC(1)$.

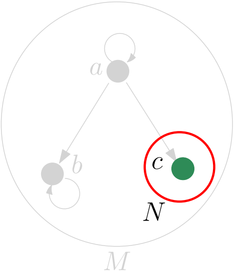
Examples (Contd.)



$M \models \phi$

- Observe: $\phi \notin PS$.

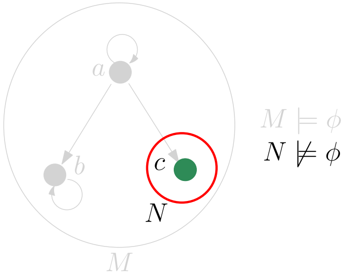
Examples (Contd.)



$M \models \phi$
$N \not\models \phi$

- Observe: $\phi \notin PS$.

Examples (Contd.)



- Observe: $\phi \notin PS$.
- Easy to see: $PSC(k) \subseteq PSC(l)$ for $k \leq l$. Then $PS \subsetneq PSC(k)$ for each $k \geq 1$.
- A Σ_2 sentence ϕ with k $\exists$ quantifiers is in $PSC(k)$.
- Converse Question: For $\phi \in PSC(k)$, is there an equivalent Σ_2 sentence having k $\exists$ quantifiers?

A Generalization of the Łoś-Tarski Theorem

Theorem 2

A FO sentence is in $PSC(k)$ iff it is equivalent to a $\exists^k\forall^*$ sentence.

- The proof is by defining a notion dual to $PSC(k)$, which we call as *preservation under k-ary covered extensions*, denoted $PCE(k)$. This notion also generalizes the classical property of preservation under extensions.
- Using the notion of *saturated structures* from classical model theory, we show that $PCE(k) \equiv \forall^k\exists^*$.
- The proof is completed via a duality lemma: $\phi \in PSC(k)$ iff $\neg\phi \in PCE(k)$.

Theorem 2 and Łoś-Tarski theorem

- The special case of $k = 0$ for Theorem 2 is exactly the Łoś-Tarski theorem for sentences.
- Theorem 2 holds in more general settings in which Łoś-Tarski theorem is already known to hold:
  - Arbitrary vocabularies (i.e. constants, predicates and functions)
  - Modulo FO theories
  - Formulae (define $\phi(\bar{x})$ to be in $PSC(k)$ if the corresponding sentence with $\bar{x}$ replaced with fresh constants is in $PSC(k)$)
- However, we are yet to have a characterization for *theories* which are in $PSC(k)$. Answering this would yield a complete subsumption of the Łoś-Tarski theorem.

Comparison with other semantic characterizations of Σ_2

- There are characterizations in the literature for Σ_2 using unions of ascending chains, unions of descending chains, 1-sandwiches, etc. However *none* of these relate the *count* of the quantifiers to any model-theoretic properties.
- All of the above characterizing notions become trivial in the finite. However, the $PSC(k)$ condition remains non-trivial in the finite – there are sentences outside of $\bigcup_{k \geq 0} PSC(k)$.
- The $PSC(k)$ condition is combinatorial in nature unlike any of the above notions.

Future Work

Over arbitrary structures:

- A syntactic characterization of theories in $PSC(k)$.

Over finite structures:

- Investigating our result over the class of coloured finite trees and multiple coloured equivalence relations (our result is true over the class of coloured finite linear orders and the class of a single coloured equivalence relation).
- Investigating our result over path graphs, tree graphs. More generally graphs of bounded degree, bounded tree-width, bounded split-width, etc.

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