Streaming String Transducers

Towards a Theory of Regular Transformations

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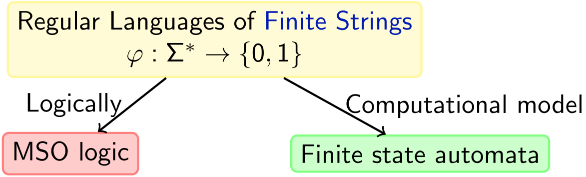
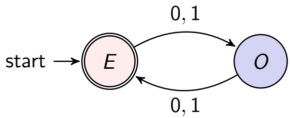
Theory of Regular Languages

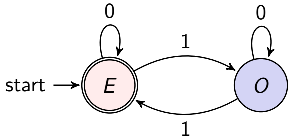


Finite State Automata

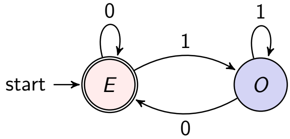
Automaton accepting strings of even length:



Automaton accepting strings with an even number of 1's:



Automaton accepting even strings (multiple of 2):



Monadic Second Order Logic (MSO) over Graphs

– The structure is

$$(\boxed{N}, \boxed{E}, \boxed{L_a, L_b, \ldots, L_k})$$

The domain (set of nodes)

Edge relation $E \subseteq N \times N$

Some unary predicates $L_a, L_b, \ldots, L_k \subseteq N$ partitioning N

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– Strings are interpreted structures: e.g. $(\{1, \ldots, 10\}, E, L_a, L_b, L_c)$

$$\begin{array}{lllllllllll}
s = & a & b & b & a & b & c & a & b & c & c \\
L_a = \{ & 1, & & & 4, & & & 7\} & & & \\
L_b = \{ & & 2, & 3, & & 5, & & & 8\} & & \\
L_c = \{ & & & & & & 6, & & & 9, & 10\}
\end{array}$$

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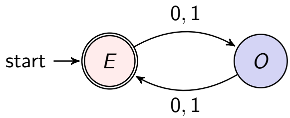
- Formulas are defined inductively:
  - first-order variables: x, y, z ranging over nodes
  - second-order variables: X, Y, Z ranging over node sets
  - Atomic formulas: $E(x, y)$, $L_a(x)$, $x = y$ and $x \in X$, ...
  - Boolean connectives: $\varphi_1 \wedge \varphi_2$, $\neg\varphi_3$, ...
  - First-order quantification: $\exists x.\varphi$
  - Second-order quantification: $\exists X.\varphi$

Examples

Set of strings with an even number of letters:



- Consider two sets of positions Even and Odd.
- Both sets are disjoint.
- First position is in Odd and the last position is in Even.
- For each position in Even the next position (if exists) is in Odd and vice-versa.

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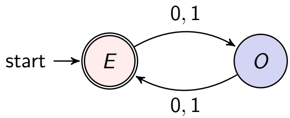

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$$
\begin{aligned}
&\exists Odd.\exists Even.\\
&(\forall x.((x \in Odd) \rightarrow \neg(x \in Even) \wedge ((x \in Even) \rightarrow \neg(x \in Odd)))\\
&\wedge First(x) \rightarrow (x \in Odd)\\
&\wedge Last(x) \rightarrow (x \in Even)\\
&\forall x \forall y((x \in Odd) \wedge E(x, y)) \rightarrow y \in Even\\
&\forall x \forall y((x \in Even) \wedge E(x, y)) \rightarrow y \in Odd).
\end{aligned}
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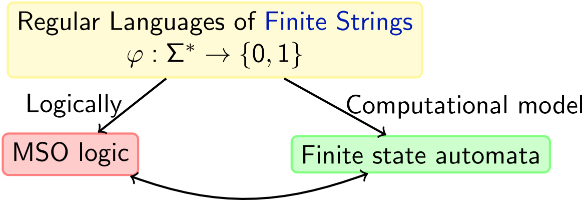

Equi-expressiveness [Büchi, 1960, Elgot, 1961, Trakhtenbrot, 1962]

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A language of finite strings is accepted by a finite state automaton iff it is MSO-definable.

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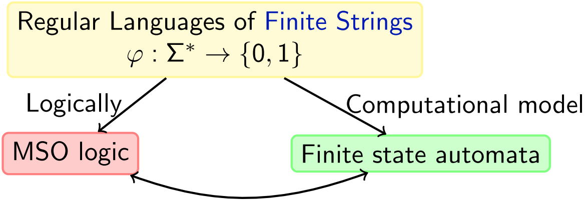

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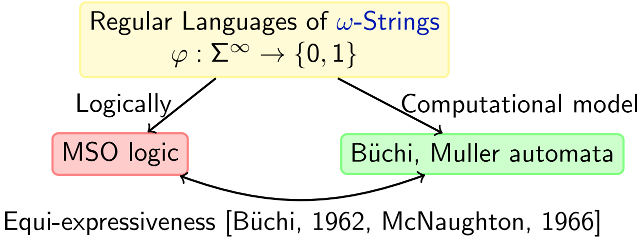
Why bother?

- new tools to solve problems in logic
- revolutionized the field of automata theory as Büchi initiated the study of equivalent finite state models for MSO over infinite strings.

Theory of Regular Languages



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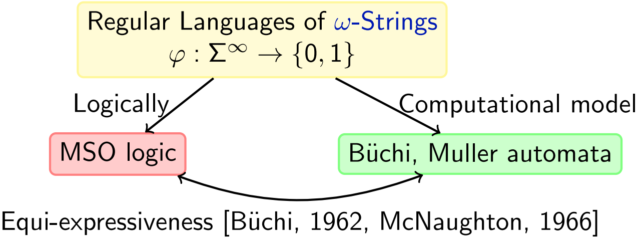

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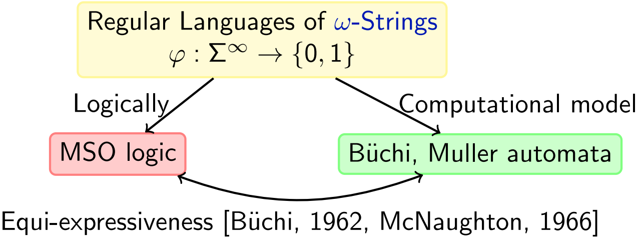

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Can we go beyond Languages!

Theory of Regular Transformations

Regular Transformations of Finite Strings
$\varphi : \Sigma^* \to \Gamma^*$

Logically

MSO logic [Courcelle, 1994]

Computational model

Finite state transducers

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(two-way) Finite state transducers

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- Unfortunately, two-way finite state transducers can not naturally be generalized with such ease

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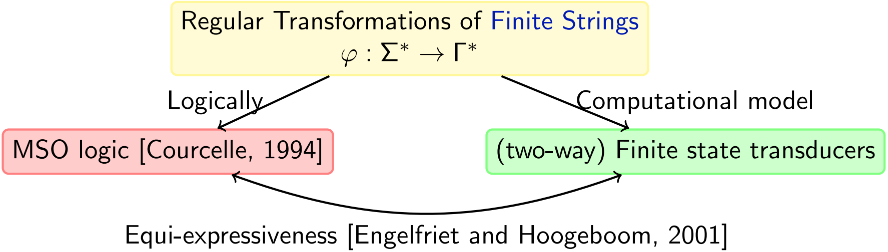

- MSO-definable transformations can be naturally extended to define transformations for more general structures
- Unfortunately, two-way finite state transducers can not naturally be generalized with such ease
- Also, it would be nice to have a one-way (streaming) transducer precisely capturing the class of MSO-definable transformations

Streaming String Transducers

- Alur and Černý introduced streaming string transducers (SSTs) to model and analyze single-pass list processing programs [Alur and Černý, 2010], e.g.
  - imperative programs manipulating heap-allocated lists
  - functional programs using tail recursion
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Theory of regular transformations

Regular Transformations of Finite Strings

Regular Transformations of Infinite Strings

Conclusion

Transformations of Finite Strings

- A transformation from Σ to Γ is a (partial) function $f : \Sigma^* \to \Gamma^*$.
- Generalizes the concept of a language $f : \Sigma^* \to \{0, 1\}$.
- Example:
  - $a^n \mapsto a^n b^n$
  - $a^n b^m \mapsto a^{2^n - 1} b^m$
  - local transformations, e.g., delete each a, repeat every b
  - reverse transformation, i.e. $a_1 a_2 \ldots a_n \mapsto a_n a_{n-1} \ldots a_1$,
  - swapping transformation, e.g. $\alpha \# \beta \mapsto \beta \# \alpha$,
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    - replace each a with b if the string contains a $\#$.
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- A transducer is an abstract machine defining a transformation.
- Transducers generalize the concept of automata
- Similar to languages, a transformation can also be defined using logic, most notably Monadic second-order logic (MSO) over finite strings.

MSO-definable Transformations

Definition (Defining Transformation using MSO)

A transformation using MSO is specified by:

- input and output alphabets;
- an MSO formula specifying the domain of the transformation;
- output string is specified using a finite number of copies of nodes of input string graph;
- the node labels are specified using MSO formulas; and
- the existence of edges between nodes of various copies is specified using MSO formulas

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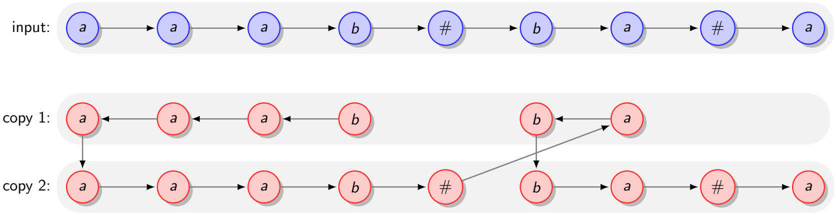
Example

Let $\Sigma = \{a, b, \#\}$. Consider a transformation $f_1 : \Sigma^* \to \Sigma^*$

$$u_1 \# u_2 \# \dots u_{n-1} \# u_n \# v \mapsto \overline{u_1} u_1 \# \dots \# \overline{u_n} u_n \# v.$$

where $\overline{u}$ is reverse of u.

MSO-definable Transformations



- $\Sigma = \Gamma = \{a, b, \#\}$, $C = \{1, 2\}$, and
- Node Label Formulas
  - $\mathrm{Label}^{c1}_{\alpha}(x) = \mathrm{Label}^{\mathrm{inp}}_{\alpha}(x) \wedge \neg\mathrm{Label}^{\mathrm{inp}}_{\#}(x) \wedge \mathrm{reach}_{\#}(x)$
  - $\mathrm{Label}^{c2}_{\alpha}(x) = \mathrm{Label}^{\mathrm{inp}}_{\alpha}(x)$
- Edge Label Formulas
  - $\mathrm{Edge}^{c1,c1}(x, y) = \mathrm{Edge}^{\mathrm{inp}}(y, x) \wedge \mathrm{Label}^{\mathrm{inp}}_{\star}(x) \wedge \mathrm{Label}^{\mathrm{inp}}_{\star}(y)$.
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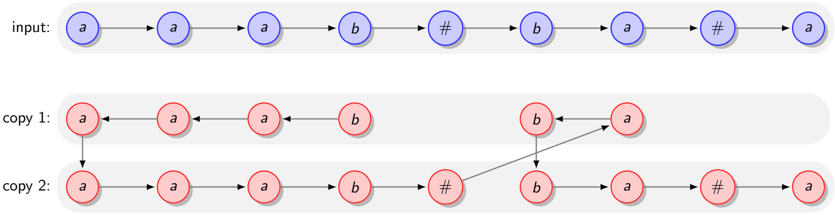
  - $\mathrm{Edge}^{1,2}(x, y) = (x = y) \wedge (\mathrm{first}(x) \vee \exists z (\mathrm{Label}^{\mathrm{inp}}_{\#}(z) \wedge \mathrm{Edge}^{\mathrm{inp}}(z, x)))$
  - $\mathrm{Edge}^{2,1}(x, y) = \mathrm{Label}^{\mathrm{inp}}_{\#}(x) \wedge \mathrm{reach}_{\#}(x) \wedge (\exists z (\mathrm{Edge}^{\mathrm{inp}}(y, z) \wedge$
    $\mathrm{Label}^{\mathrm{inp}}_{\#}(z))) \wedge (\forall z ((\mathrm{path}(x, z) \wedge \mathrm{path}(z, y)) \rightarrow \neg\mathrm{Label}^{\mathrm{inp}}_{\#}(z)))$

MSO-definable Transformations



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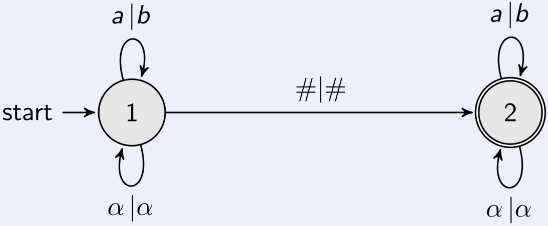
Regular Transformations

Which transducers accept same class of transformations?

Deterministic Generalized Sequential Machines

Example: For all strings containing a $\#$, replace all a with b.

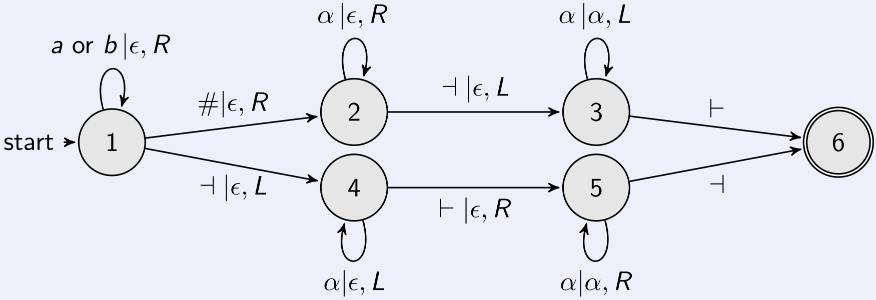


†Here α stands for any symbol other than a.

- Extend finite automata with output
- Can express local transformations
- Can not express reverse, swap, or regular look-ahead
- Non-deterministic variants can express regular look-ahead

2-Way Deterministic Finite State Transducers

Example: $u \mapsto$ if u contains a $\#$ then $\overline{u}$ else u



†Here α stands for any symbol except end markers.

- Extend two-way finite automata with output
- Allowing transitions based on regular look-ahead do not increase expressiveness (Chytil and Jakl [1977])
- Two-way finite-state transducers capture the same class of MSO-definable transformations (Engelfriet and Hoogeboom [2001])

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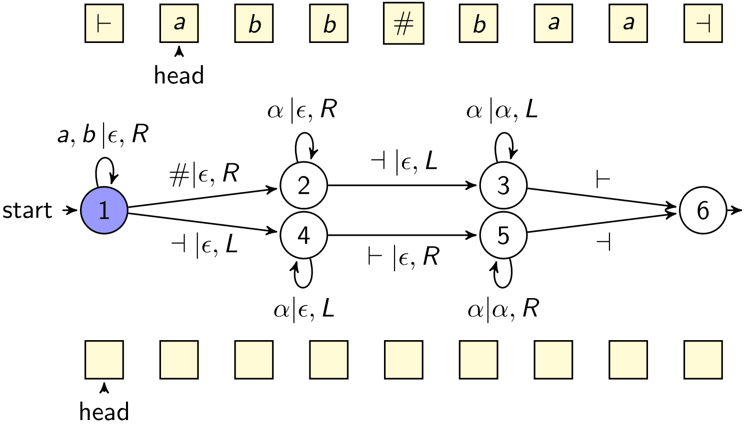

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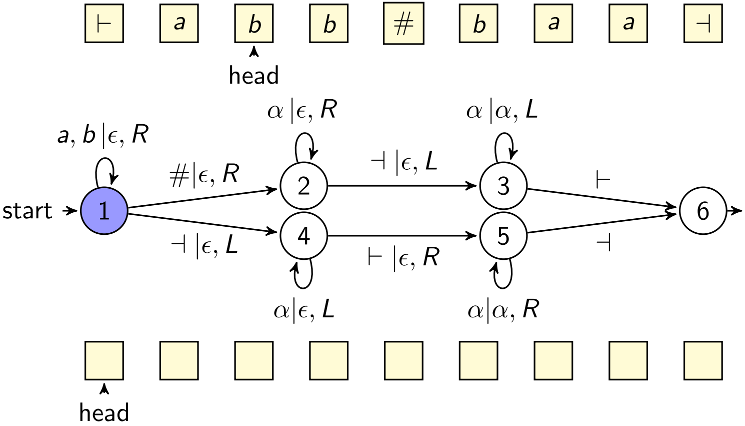

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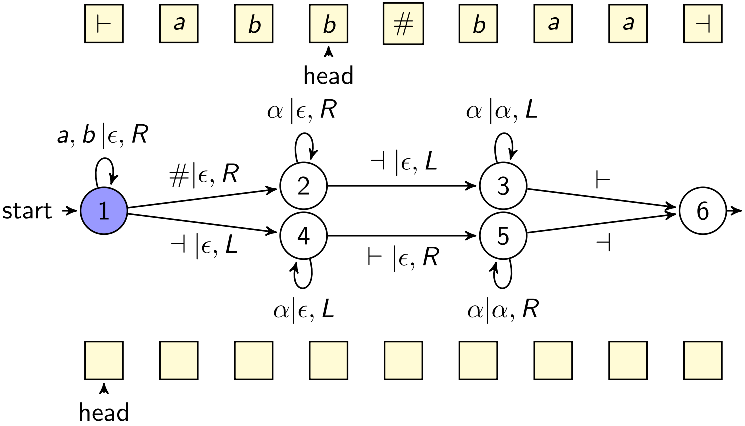

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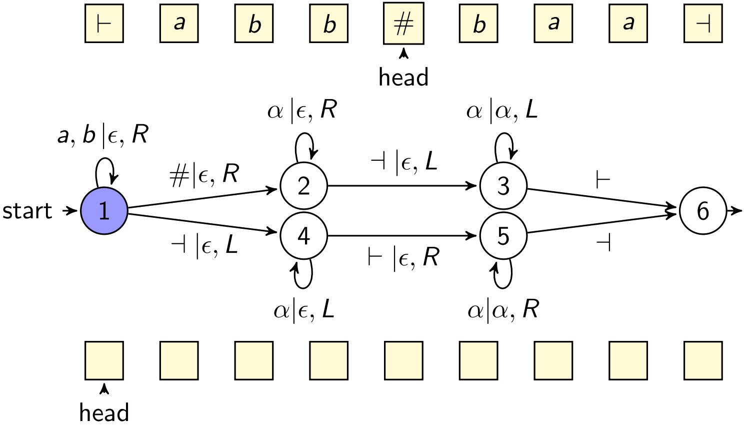

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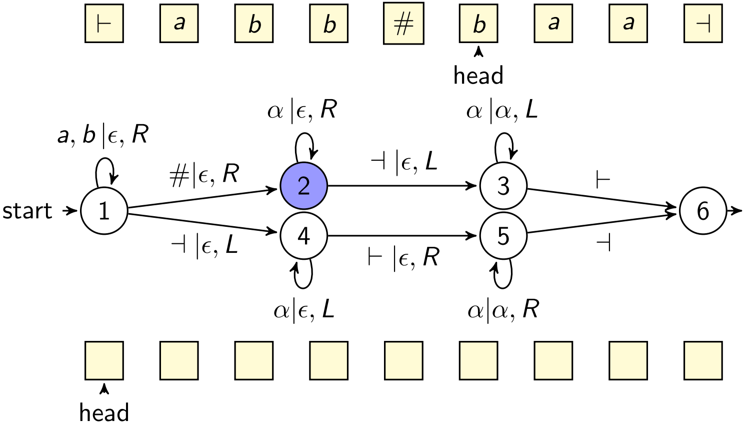

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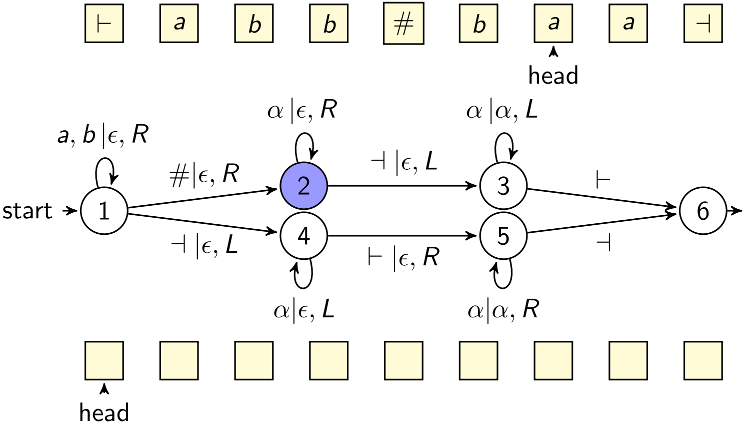

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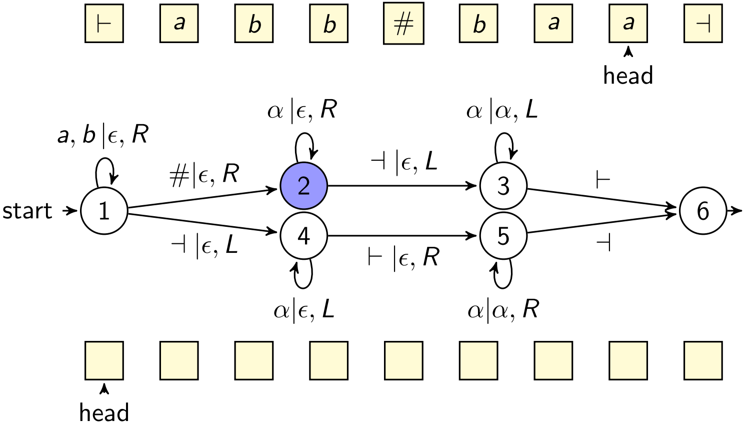

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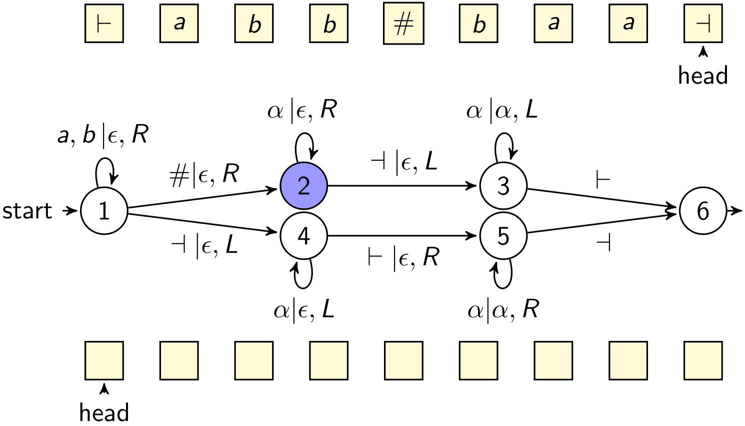

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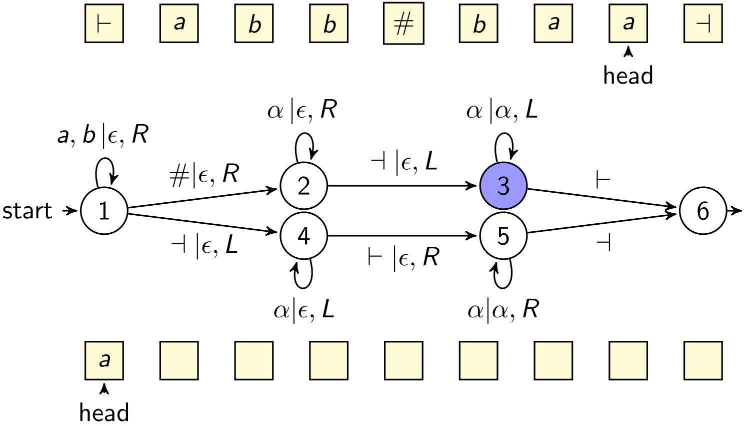

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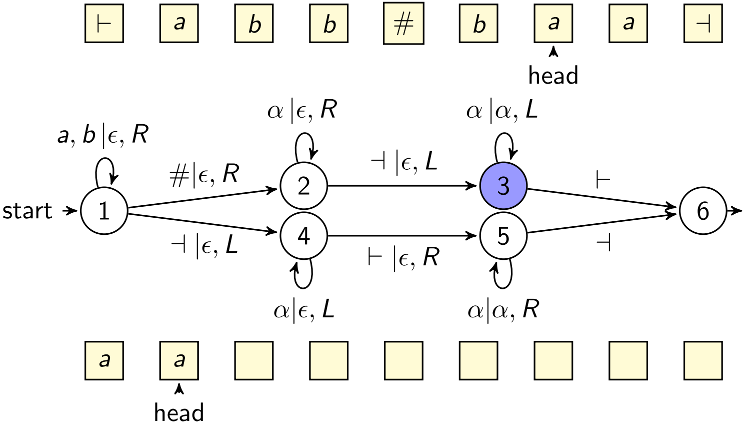

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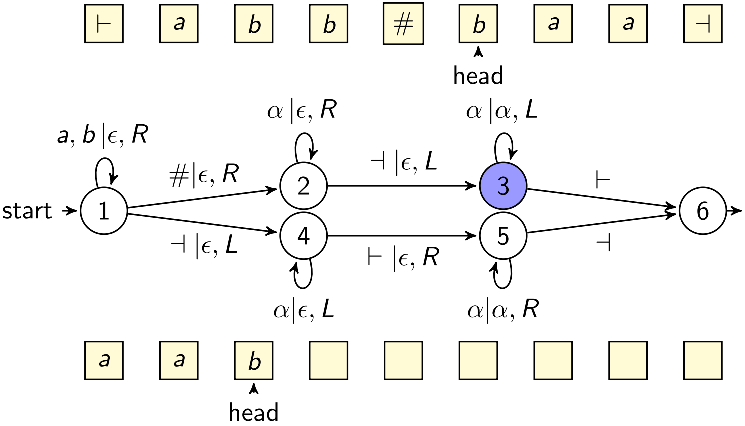

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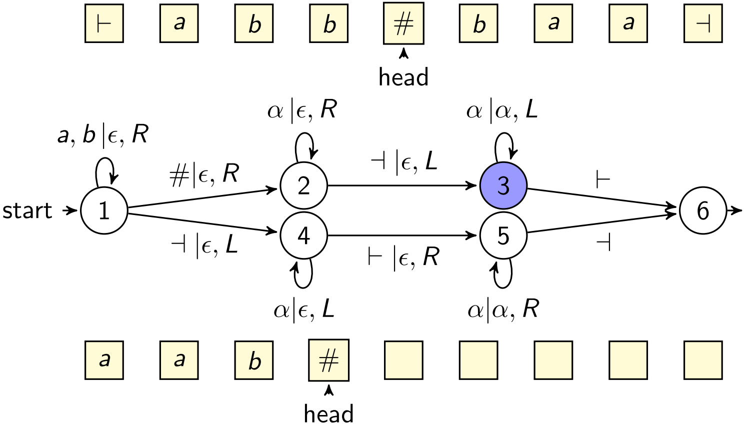

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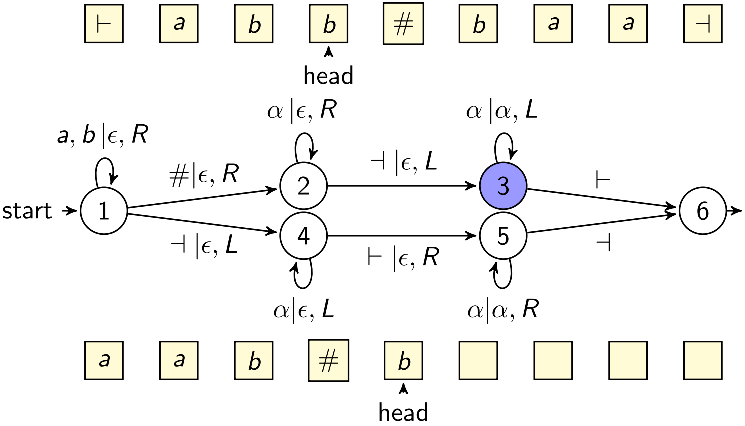

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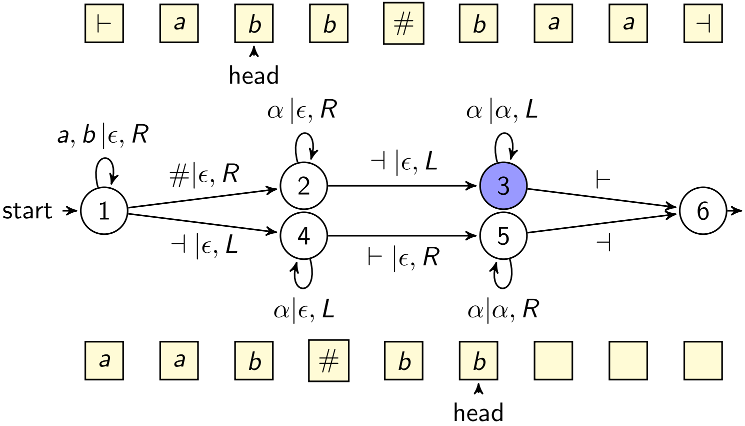

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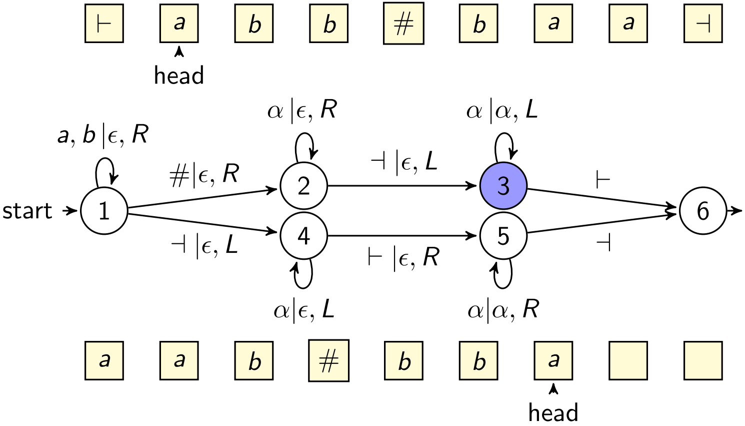

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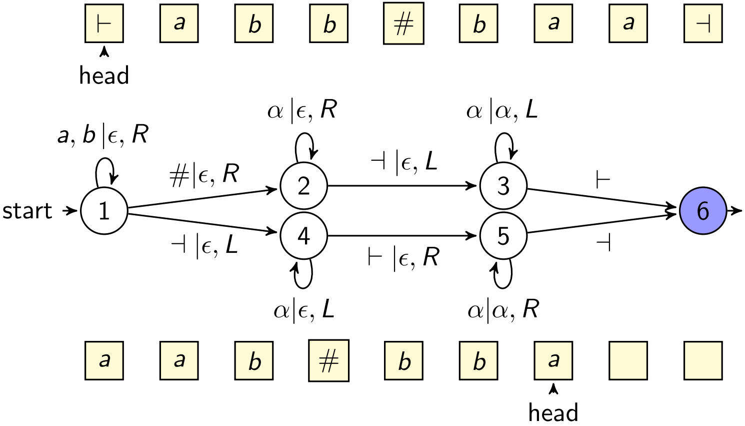

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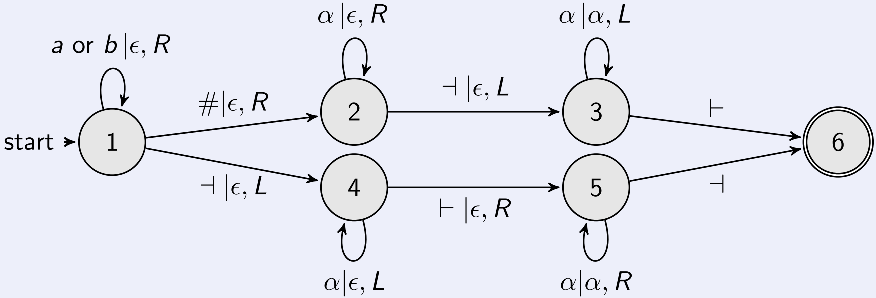
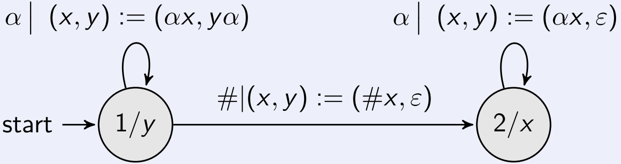


†Here α stands for any symbol except end markers.

- Extend two-way finite automata with output
- Allowing transitions based on regular look-ahead do not increase expressiveness (Chytil and Jakl [1977])
- Two-way finite-state transducers capture the same class of MSO-definable transformations (Engelfriet and Hoogeboom [2001])

Transducers: Streaming String Transducers

Example: $u \mapsto$ if u contains a $\#$ then $\overline{u}$ else u



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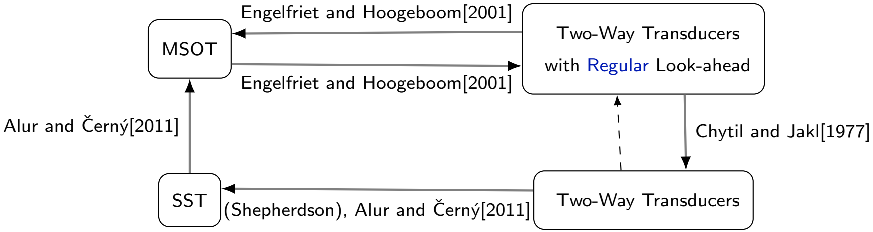
- Extend deterministic finite-state automata with string variables
- String variables are updated in a copyless fashion
- Output is given as a function of states to copyless concatenation of string variables

Expressiveness of Streaming String Transducers

Theorem ([Alur and Černý, 2011])

A transformation of finite strings is accepted by a streaming string transducer iff it is MSO-definable.



Properties of Regular Transformations

- Characterized by
  - MSO,
  - (deterministic) two-way finite-state transducers, and
  - (deterministic) streaming string transducers.
- They are closed under sequential composition
- Equivalence problem, deciding the equivalence of two regular transformations, is decidable.
- Type checking problem, deciding whether image of a given regular set I under a regular transformation T is contained in another given regular set O i.e. $T(I) \subseteq O$, is decidable.
- Both problems are in PSPACE for streaming-string transducers [Alur and Černý, 2011]

Transformations of Infinite Strings

- A transformation from Σ to Γ is a (partial) function $f : \Sigma^\omega \to \Gamma^\omega$.
- Generalizes the concept of an ω-language $f : \Sigma^\omega \to \{0, 1\}$.
- Example:
  - $a^n \#^\omega \mapsto a^n b^n \#^\omega$
  - $a^n b^\omega \mapsto a^{2^n - 1} b^\omega$
  - local transformations, e.g., delete each a, repeat every b
  - reverse transformation, i.e. $a_1 a_2 \ldots a_n \# u \mapsto a_n a_{n-1} \ldots a_1 \# u$,
  - swapping transformation, e.g. $\alpha \# \beta \# u \mapsto \beta \# \alpha \# u$,
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    - replace each a with b if the string contains a $\#$.
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- MSO on infinite strings can be used to define transformations on infinite strings [Courcelle, 1994]

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- MSO on infinite strings can be used to define transformations on infinite strings [Courcelle, 1994]
- What classes of **finite-state transducers** have equal expressive power?
- What **decision problems** about MSO-definable transformations of infinite strings can be solved?

MSO-definable Transformations

Definition (Defining Transformation using MSO)

A transformation using MSO is specified by:

- input and output alphabets;
- an MSO formula specifying the domain of the transformation;
- output string is specified using a finite number of copies of nodes of input string graph;
- the node labels are specified using MSO formulas; and
- the existence of edges between nodes of various copies is specified using MSO formulas

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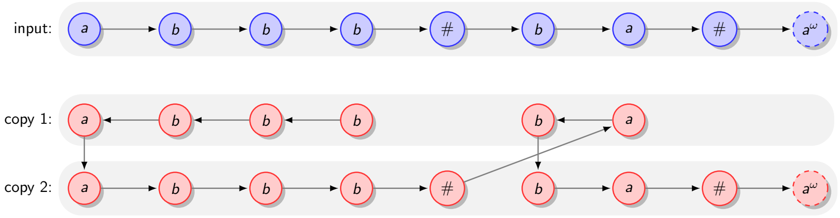
Example

Let $\Sigma = \{a, b, \#\}$. Consider a transformation $f_2 : \Sigma^\omega \rightarrow \Sigma^\omega$

$$u_1 \# u_2 \# \ldots u_{n-1} \# u_n \# v \mapsto \overline{u_1} u_1 \# \ldots \# \overline{u_n} u_n \# v.$$

where $\overline{u}$ is reverse of u and $v \in \{a, b\}^\omega$.

MSO-definable Transformations



- $\Sigma = \Gamma = \{a, b, \#\}$, $C = \{1, 2\}$, and
- Node Label Formulas
  - $\mathrm{Label}^{c1}_{\alpha}(x) = \mathrm{Label}^{\mathrm{inp}}_{\alpha}(x) \wedge \neg\mathrm{Label}^{\mathrm{inp}}_{\#}(x) \wedge \mathrm{reach}_{\#}(x)$
  - $\mathrm{Label}^{c2}_{\alpha}(x) = \mathrm{Label}^{\mathrm{inp}}_{\alpha}(x)$
- Edge Label Formulas
  - $\mathrm{Edge}^{c1,c1}(x, y) = \mathrm{Edge}^{\mathrm{inp}}(y, x) \wedge \mathrm{Label}^{\mathrm{inp}}_{\star}(x) \wedge \mathrm{Label}^{\mathrm{inp}}_{\star}(y)$.
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  - $\mathrm{Edge}^{1,2}(x, y) = (x = y) \wedge (\mathrm{first}(x) \vee \exists z(\mathrm{Label}^{\mathrm{inp}}_{\#}(z) \wedge \mathrm{Edge}^{\mathrm{inp}}(z, x)))$
  - $\mathrm{Edge}^{2,1}(x, y) = \mathrm{Label}^{\mathrm{inp}}_{\#}(x) \wedge \mathrm{reach}_{\#}(x) \wedge (\exists z(\mathrm{Edge}^{\mathrm{inp}}(y, z) \wedge$
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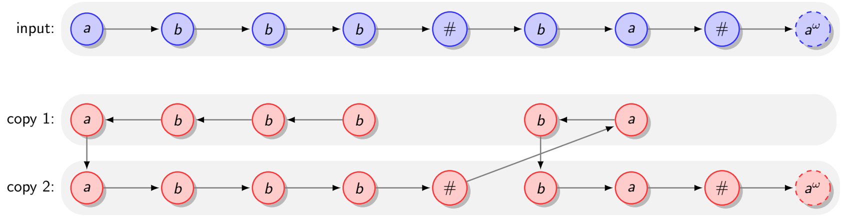
MSO-definable Transformations



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MSO-definable Transformations

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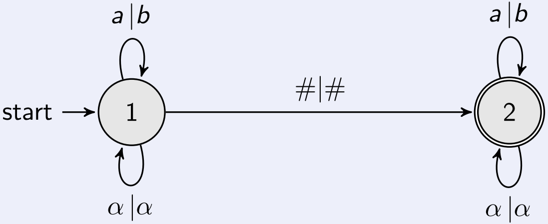
Regular Transformations on Infinite Strings

Which transducers accept the same class of transformations?

Deterministic Generalized Sequential Machines

Example: For all strings containing a $\#$, replace all a with b.

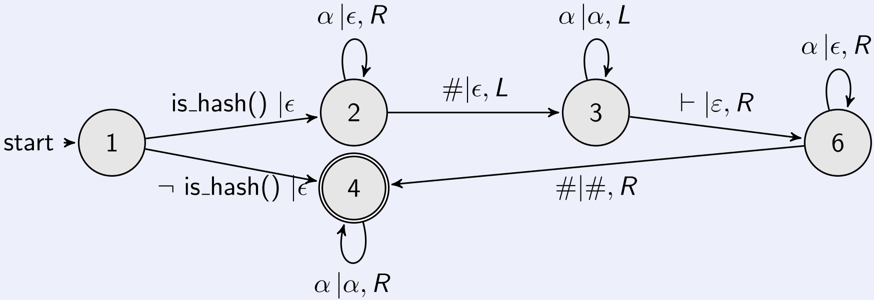


†Here α stands for any symbol other than a.

- Extend Muller automata with output
- Can express local transformations
- Can not express transformations such as reverse or swap

2-Way Transducers with Look-Ahead

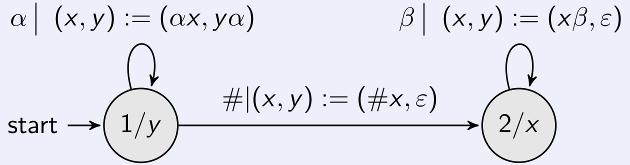
Example: Reverse the sub-string before the first



- Extend two-way Muller automata with output
- Allowing ω-regular look-ahead increases expressiveness
- Two-way finite-state transducers with ω-regular look-ahead capture the same class of transformations as MSO.

SSTs with Muller Acceptance Condition

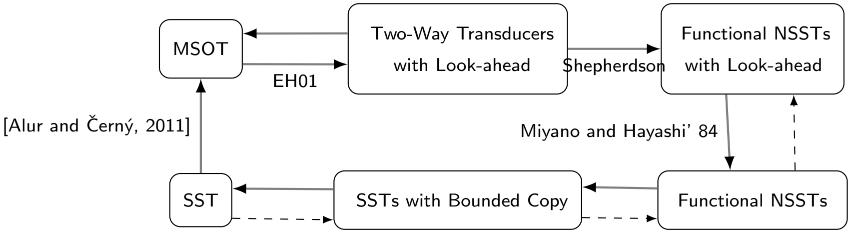
Example: Reverse the sub-string before the first



[†]Here α is any symbol except #, while β is any symbol.

- Extend Muller automata with string variables
- String variables are updated in a copyless fashion
- Output is given as a function of set of states to copyless concatenation of string variables
- We enforce syntactic restrictions that ascertain that output string is always an infinite string

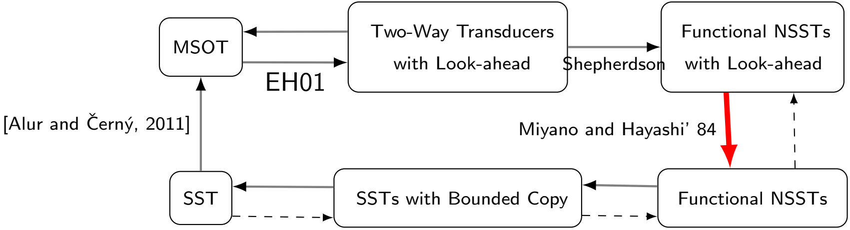
Expressiveness of Streaming String Transducers



Proof Sketch

Theorem

A transformation of infinite strings is accepted by a streaming string transducer iff it is MSO-definable.



- simulate all look-aheads in parallel
- look-ahead $\sim$ universal transitions in an alternating Muller automaton
- use Miyano-Hayashi like construction to remove universality

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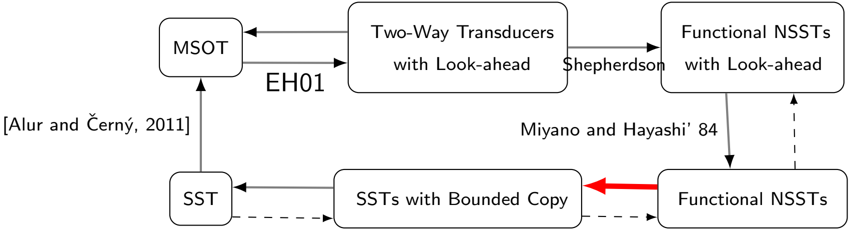

- simulate all runs in parallel
- functionality ⇒ at most $|Q|$ runs have to be simulated in parallel
- use $|Q|$ copies of each variable $x \in X$
- may introduce variable copy

Proof Sketch

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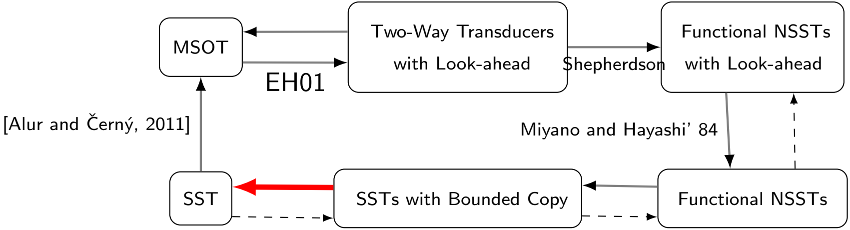

- most technical result
- based on the notion of dependency graphs
- states are sufficient abstractions of dependency graphs

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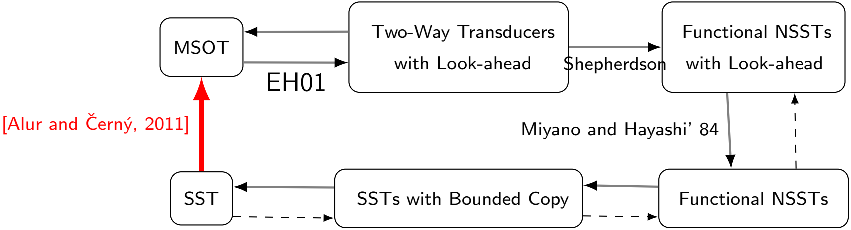

- simple extension of the finite string case
- uses two domain copies for each variable

Equivalence Problem

Theorem

Equivalence problem is decidable in PSPACE for streaming-string transducers on infinite strings.

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T_1 and T_2 are inequivalent iff $dom(T_1) \neq dom(T_2)$ **or**

$dom(T_1) = dom(T_2)$ and $\exists u \in dom(T_1)$, $\exists i \geq 0$ such that $T_1(u)[i] \neq T_2(u)[i]$

1. domain equivalence can be checked in PSPACE.
2. if domains are equivalent, then check existence of u
   - reduction to emptiness of reversal-bounded counter machines (NLogSpace, Ibarra)
   - product construction to simulate runs of T_1 and T_2 on the same inputs
   - guess a position i and check that there is a mismatch
   - as outputs are not produced synchronously, counters are used to retrieve the letters at position i in both outputs
   - construction ensures that finite runs can be extended to infinite accepting runs that do not modify the letters at position i

Type-Checking Problem

Theorem

Type-checking, deciding whether image of a given regular set I under a regular transformation T is contained in another given regular set O i.e. $T(I) \subseteq O$, is decidable in PSPACE for streaming-string transducers on infinite strings.

- Check whether T is defined for all strings of u, i.e. $\mathrm{dom}(T) \subseteq I$.
- A Muller automaton recognizing the domain of T can be constructed in linear time, and therefore $I \subseteq \mathrm{dom}(T)$ can be checked in PSPACE.

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- Next we check the language $L = \{u \in \Sigma^\omega \mid u \in I, T(u) \notin O\}$ for emptiness.
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- The language L can be defined by a Muller automaton A_L that simulates automaton A_I and T on the input string, and $A_{\overline{O}}$ on the output of T.
- This can be done by computing functions τ such that for all states q of $A_{\overline{O}}$ and all variables $x \in X$, $\tau(q, x)$ is the state of $A_{\overline{O}}$ after evaluating the current value of x, starting from state q.
- The size of A_L is exponential in A_I, A_O and T, and its emptiness can be decided in PSPACE.

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Corollary

Equivalence of MSO-transducers on infinite strings is decidable.

Regular Transformations of Finite Strings

Regular Transformations of Infinite Strings

Conclusion

Summary

- Introduction of streaming string transducers renewed the interest in the study of regular transformations
- Streaming string transducers naturally extend from strings to more general structures, while conserving MSO equivalence.
- Streaming-string transducer models are robust: closed under bounded copy, functional nondeterminism, and regular look-ahead.
- Important verification problems like functional equivalence and pre/post condition type-checking are decidable for streaming string transducers.

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Thank You!

Alur, A. and Černý, P. (2011).
Streaming transducers for algorithmic verification of single-pass list-processing programs.
In *POPL*, pages 599–610.

Alur, R. and Černý, P. (2010).
Expressiveness of streaming string transducers.
In *FSTTCS*, volume 8, pages 1–12.

Alur, R. and D'Antoni, L. (2012).
Streaming tree transducers.
In *ICALP (2)*, pages 42–53.

Alur, R., D'Antoni, L., Deshmukh, J. V., Raghothaman, M., and Yuan, Y. (2013a).
Regular functions and cost register automata.
In *LICS*.

Alur, R., Durand-Gasselin, A., and Trivedi, A. (2013b).
From monadic second-order definable string transformations to transducers.
In *LICS*.

Alur, R., Filiot, E., and Trivedi, A. (2012).

Regular transformations of infinite strings.
In *LICS*, pages 65–74.

Büchi, J. R. (1960).
Weak second-order arithmetic and finite automata.
Zeitschrift für Mathematische Logik und Grundlagen der Mathematik, 6(1–6):66–92.

Büchi, J. R. (1962).
On a decision method in restricted second-order arithmetic.
In *Int. Congr. for Logic Methodology and Philosophy of Science*, pages 1–11. Standford University Press, Stanford.

Courcelle, B. (1994).
Monadic second-order definable graph transductions: a survey.
Theoretical Computer Science, 126(1):53–75.

Elgot, C. C. (1961).
Decision problems of finite automata design and related arithmetics.
In Transactions of the American Mathematical Society, 98(1):21–51.

Engelfriet, J. and Hoogeboom, H. J. (2001).
MSO definable string transductions and two-way finite-state transducers.
ACM Trans. Comput. Logic, 2:216–254.

McNaughton, R. (1966).
Testing and generating infinite sequences by a finite automaton.
Inform. Contr., 9:521–530.

Rabin, M. O. (1969).
Decidability of second-order theories and automata on infinite trees.
Transactions of the American Mathematical Society, 1(35).

Thomas, W. (1995).
On the synthesis of strategies in infinite games.
In *STACS*, volume 900 of *LNCS*, pages 1–13. Springer.

Trakhtenbrot, B. A. (1962).
Finite automata and monadic second order logic.
Siberian Mathematical Journal, 3:101–131.