

# On game logics

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Formal Methods Update Meeting 2008

TRDDC, Pune

July 18, 2008

# What game logic is all about

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- uniform study

# The plan today

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- Parikh and Pauly

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- Bisimulation
- Game Algebra
- van Benthem's representation
- determined vs. non-determined
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- Concurrent Dynamic Game Logic
- Current focus

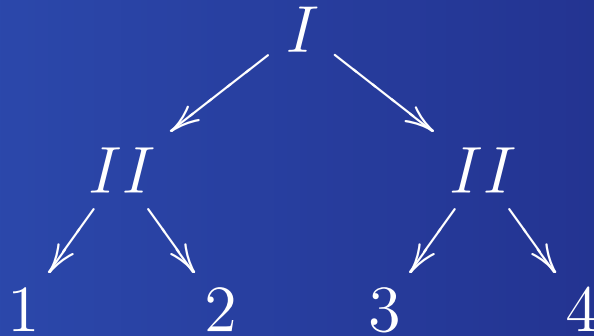
# Forcing Relations

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$s\rho_G^i X$  : player  $i$  has a strategy for playing game  $G$  from state  $s$  onwards, whose resulting states are always in the set  $X$ .

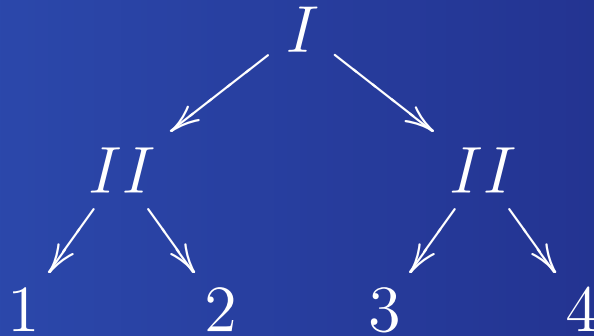
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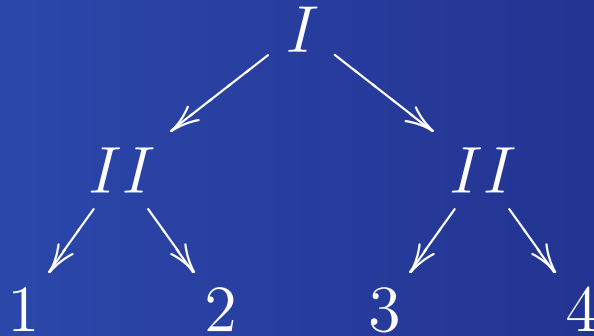
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- Consistency: If  $s\rho_G^I Y$  and  $s\rho_G^{II} Z$ , then  $Y$  and  $Z$  overlap.
- Determinacy: If it is not the case that  $s\rho_G^I X$ , then,  $s\rho_G^{II} S - X$ , and the same for  $II$  vis-a-vis  $I$ , where  $S$  denotes the total set of states.

# Composite Game Structures

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*Choice*  $(G \cup G')$ ,

*Dual*  $(G^d)$ ,

*Sequential composition*  $(G; G')$ ,

*Iteration*  $(G^*)$

# Composite Game Structures

$$s\rho_{G \cup G'}^I X \quad \text{iff} \quad s\rho_G^I X \text{ or } s\rho_{G'}^I X$$

$$s\rho_{G \cup G'}^{II} X \quad \text{iff} \quad s\rho_G^{II} X \text{ and } s\rho_{G'}^{II} X$$

$$s\rho_{G^d}^I X \quad \text{iff} \quad s\rho_G^{II} X$$

$$s\rho_{G^d}^{II} X \quad \text{iff} \quad s\rho_G^I X$$

$$s\rho_{G;G'}^i X \quad \text{iff} \quad \exists Z : s\rho_G^i Z \text{ and for all } z \in Z, z\rho_{G'}^i X$$

$$s\rho_{G^*}^I X \quad \text{iff} \quad s \in \mu Y. X \cup \{z \mid z\rho_G^I Y\}$$

$$s\rho_{G^*}^{II} X \quad \text{iff} \quad s \in \nu Y. X \cup \{z \mid z\rho_G^{II} Y\}$$



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# DGL

Language:

$$\gamma := g \mid \phi? \mid \gamma; \gamma \mid \gamma \cup \gamma \mid \gamma^* \mid \gamma^d$$
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Game Model:  $\mathcal{M} = (S, \{\rho_g^I \mid g \in \Gamma\}, V)$

Semantics:

$\mathcal{M}, s \models \langle \gamma \rangle \phi$  iff there exists  $X : s \rho_\gamma^I X$  and

$\forall x \in X : \mathcal{M}, x \models \phi$

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a) all propositional tautologies and inference rules

b) if  $\vdash \phi \rightarrow \psi$  then  $\vdash \langle \gamma \rangle \phi \rightarrow \langle \gamma \rangle \psi$

c) if  $\vdash (\phi \vee \langle \gamma \rangle \psi) \rightarrow \psi$  then  $\vdash \langle \gamma^* \rangle \phi \rightarrow \psi$

d) reduction axioms:

$$\langle \alpha \cup \beta \rangle \phi \leftrightarrow \langle \alpha \rangle \phi \vee \langle \beta \rangle \phi$$

$$\langle \alpha; \beta \rangle \phi \leftrightarrow \langle \alpha \rangle \langle \beta \rangle \phi$$

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(Pauly, 2001)

Complete axiomatization of  $DGL!!$

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- (Berwanger, Grädel and Lenzi, 2006) The expressive power of *DGL* is strictly less than that of modal mu-calculus.

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- A *DGL* formula  $\varphi$  is invariant for bisimulation if for all game models,  $\mathcal{M}$  and  $\mathcal{M}'$ ,  $s \Leftrightarrow s'$  implies,  $\mathcal{M}, s \models \varphi \Leftrightarrow \mathcal{M}', s' \models \varphi$ .

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- A *DGL*-game  $\gamma$  is safe for bisimulation if for all game models,  $\mathcal{M}$  and  $\mathcal{M}'$ ,  $s \Leftrightarrow s'$  implies,
  - (1) if  $s \rho_\gamma^i X$ , then  $\exists X' \subseteq S'$ , such that  $s' \rho_\gamma^{i'} X'$ , and  $\forall x' \in X', \exists x \in X, x \Leftrightarrow x'$ .
  - (2) if  $s' \rho_\gamma^{i'} X'$ , then  $\exists X \subseteq \mathcal{P}(S)$ , such that  $s \rho_\gamma^i X$ , and  $\forall x \in X, \exists x' \in X' : x \Leftrightarrow x'$ .

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Game expressions  $G$  and  $G'$  are identical if their interpretations in any game model give the same forcing relations.

# Game Algebra

$$x \vee x \approx x \qquad x \wedge x \approx x \qquad (G1)$$

$$x \vee y \approx y \vee x \qquad x \wedge y \approx y \wedge x \qquad (G2)$$

$$x \vee (y \vee z) \approx (x \vee y) \vee z \qquad x \wedge (y \wedge z) \approx (x \wedge y) \wedge z \qquad (G3)$$

$$x \vee (y \wedge z) \approx x \qquad x \wedge (y \vee z) \approx x \qquad (G4)$$

$$x \vee (y \wedge z) \approx (x \vee y) \wedge (x \vee z) \qquad x \wedge (y \vee z) \approx (x \wedge y) \vee (x \wedge z) \qquad (G5)$$

$$\neg \neg x \approx x \qquad (G6)$$

$$\neg(x \vee y) \approx \neg x \wedge \neg y \qquad \neg(x \wedge y) \approx \neg x \vee \neg y \qquad (G7)$$

$$(x \diamond y) \diamond z \approx x \diamond (y \diamond z) \qquad (G8)$$

$$(x \vee y) \diamond z \approx (x \diamond z) \vee (y \diamond z) \qquad (x \wedge y) \diamond z \approx (x \diamond z) \wedge (y \diamond z) \qquad (G9)$$

$$\neg x \diamond \neg y \approx \neg(x \diamond y) \qquad (G10)$$

$$y \preceq z \rightarrow x \diamond y \preceq x \diamond z \qquad (G11)$$

$s \preceq t$  is an abbreviation of the equation  $s \vee t \approx t$ , and  $\wedge$  denotes the dual game of  $\vee$ .

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Conjectured by (van Benthem, 1999), completeness proved by (Venema, 2003) and (Goranko, 2003)

# First order Evaluation Games

Two players, Verifier  $V$  and Falsifier  $F$ , dispute the truth of a formula  $\phi$  in some model  $\mathcal{M}$ . The game starts from a given assignment  $s$  sending variables to objects in the domain of some given model. Verifier claims that the formula is true in  $\mathcal{M}$ , Falsifier claims that it is false. The rules of this game  $eval(\phi, \mathcal{M}, s)$  are defined as follows:

- If  $\phi$  is an atom,  $V$  wins if the atom is true, and  $F$  wins if it is false.
- For formulas  $\phi \vee \psi$ ,  $V$  chooses a disjunct to continue with.
- For formulas  $\phi \wedge \psi$ ,  $F$  chooses a conjunct to continue with.
- With negation  $\neg\phi$ , the two players switch roles.
- For an existential quantifier  $\exists x\psi$ ,  $V$  chooses an object  $d$  in  $\mathcal{M}$ , and play continues w.r.t  $\phi$  and the new assignment  $s[x:=d]$ .
- For a universal quantifier  $\forall x\psi$ ,  $F$  chooses an object  $d$  in  $\mathcal{M}$ , and play continues w.r.t  $\phi$  and the new assignment  $s[x:=d]$ .

# Logic Games vs. Game Logics

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First-order evaluation games are a special case of *DGL*, where the atomic game is: variable-to-value reassignment for quantifiers by themselves.

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What about the converse?

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- Any two families  $F_1$  and  $F_2$  of subsets of some set  $S$  satisfying the three earlier conditions *monotonicity*, *consistency*, and *determinacy* are the powers of players at the root of some two-step extensive game.

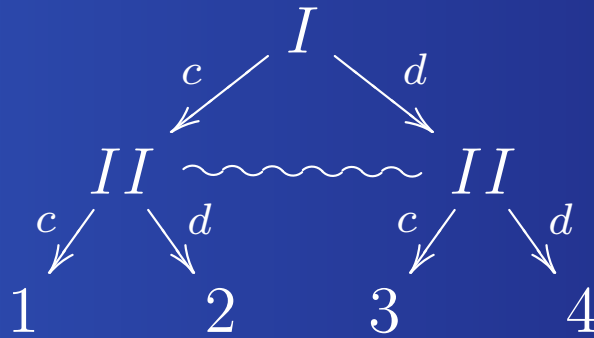
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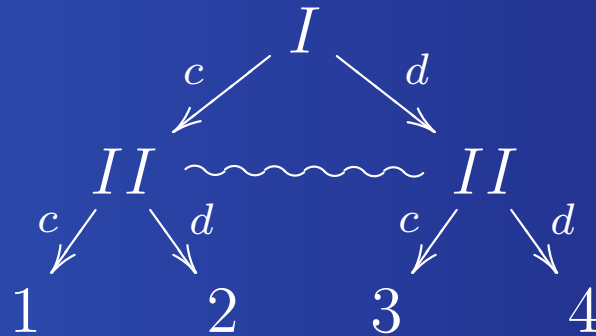
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- There is a faithful embedding of *DGL* into the game logic of first-order evaluation games.  
(van Benthem, 2003)

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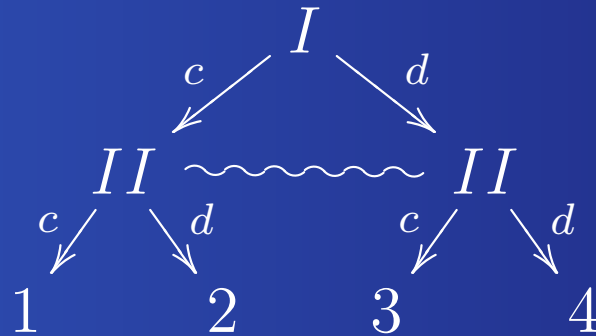


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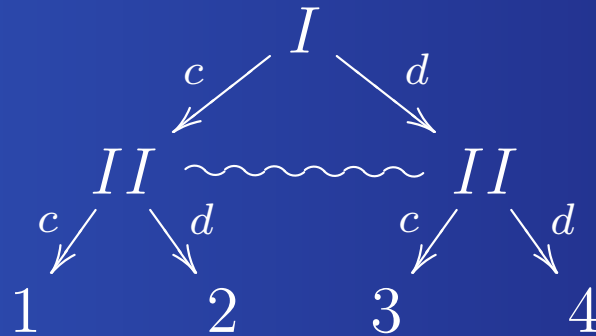
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$II$ 's powers :  $\{1, 3\}, \{2, 4\}$ .

Neither  $\{2,3\}$  is a power of  $I$ , nor  $\{1, 4\}$  of  $II$ .

Non-determinacy creeps in!



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(van Eijck and Verbrugge, 2008)

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- Sequence: Either  $s\rho_G^I S$  or  $s\rho_G^{II} S$ .

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$$s\rho_{G^d}^{II} X \quad \text{iff} \quad s\rho_G^I X$$

$$s\rho_{G;G'}^i X \quad \text{iff} \quad \exists Z : s\rho_G^i Z \text{ and for all } z \in Z, z\rho_{G'}^i X$$

$$s\rho_{G^*}^I X \quad \text{iff} \quad s \in \mu Y.X \cup \{z \mid z\rho_G^I Y\}$$

$$s\rho_{G^*}^{II} X \quad \text{iff} \quad s \in \nu Y.X \cup \{z \mid z\rho_G^{II} Y\}$$

# Complete Axiomatization

# Complete Axiomatization

- All instantiations of propositional tautologies and inference rules.
- The monotonicity rule for the basic game modalities:  
if  $\vdash \phi_1 \rightarrow \phi_2$  then  $\vdash \langle g, i \rangle \phi_1 \rightarrow \langle g, i \rangle \phi_2$ .
- The consistency axiom for the basic game modalities:  
 $\vdash \langle g, I \rangle \phi \rightarrow \neg \langle g, II \rangle \neg \phi$ .
- The sequence axiom for the basic game modalities:  
 $\vdash \langle g, I \rangle \top \vee \langle g, II \rangle \top$ .
- The least fixpoint rule for  $I$ -iteration:  
if  $\vdash (\phi_1 \vee \langle \gamma, I \rangle \phi_2) \rightarrow \phi_2$  then  $\vdash \langle \gamma^*, I \rangle \phi_1 \rightarrow \phi_2$ .
- The greatest fixpoint rule for  $II$ -iteration:  
if  $\vdash \phi_1 \rightarrow (\phi_2 \wedge \langle \gamma, II \rangle \phi_1)$  then  $\vdash \phi_1 \rightarrow \langle \gamma^*, II \rangle \phi_2$ .

# Complete Axiomatization

## Reduction axioms:

$$\vdash \langle \gamma_1 \cup \gamma_2, I \rangle \phi \leftrightarrow \langle \gamma_1, I \rangle \phi \vee \langle \gamma_2, I \rangle \phi.$$

$$\vdash \langle \gamma_1 \cup \gamma_2, II \rangle \phi \leftrightarrow \langle \gamma_1, II \rangle \phi \wedge \langle \gamma_2, II \rangle \phi.$$

$$\vdash \langle \gamma^d, i \rangle \phi \leftrightarrow \langle \gamma, \bar{i} \rangle \phi.$$

$$\vdash \langle \gamma_1; \gamma_2, i \rangle \phi \leftrightarrow \langle \gamma_1, i \rangle \langle \gamma_2, i \rangle \phi.$$

$$\vdash \langle \phi_1?, I \rangle \phi_2 \leftrightarrow \phi_1 \wedge \phi_2.$$

$$\vdash \langle \phi_1?, II \rangle \phi_2 \leftrightarrow \neg \phi_1 \wedge \phi_2.$$

## Unfolding axioms:

$$\vdash \langle \gamma^*, I \rangle \phi \leftrightarrow \phi \vee \langle \gamma; \gamma^*, I \rangle \phi.$$

$$\vdash \langle \gamma^*, II \rangle \phi \leftrightarrow \phi \wedge \langle \gamma; \gamma^*, II \rangle \phi.$$

# Non-determinacy in test games

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$$s \rho_{p?;q?}^I X \quad \text{iff} \quad s \in \llbracket p \rrbracket \cap \llbracket q \rrbracket \cap X$$

$$s \rho_{p?;q?}^{II} X \quad \text{iff} \quad s \in (S - \llbracket p \rrbracket) \cap (S - \llbracket q \rrbracket) \cap X$$

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The two-part game over  $p?;q?$  is a win for  $I$  if both  $p$  and  $q$  happen to be true, a win for  $II$  if both happen to be false, and a draw otherwise.

# Concurrent games

# Simultaneous/Parallel Games

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Prisoner's Dilemma.

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# Branching Quantifiers

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$$\begin{array}{c} \forall x \exists y \\ \quad \diagdown \\ \quad \quad Rxyz \\ \quad \diagup \\ \forall z \exists u \end{array}$$

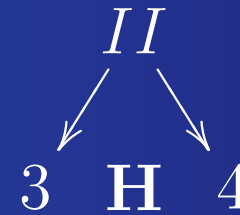
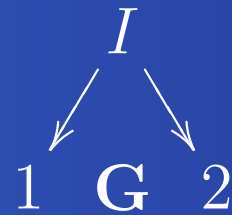
# Forcing Relation for Product Games

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$$s \rho_g^i X, s \in S, \text{ and } X \subseteq \mathcal{P}(S)$$

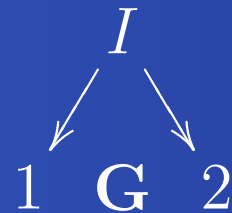
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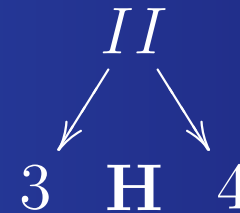
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$s \rho_g^i X, s \in S, \text{ and } X \subseteq \mathcal{P}(S)$



*I*'s power :  $\{\{1\}\}, \{\{2\}\}$ .

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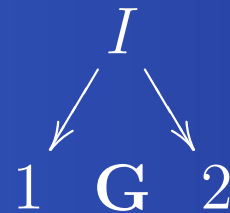


*I*'s power :  $\{\{3\}, \{4\}\}$ .

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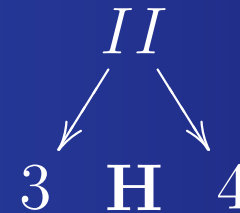
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Each outcome state is a set read '**conjunctively**', but players have choices leading to sets of these read '**disjunctively**' as in *DGL*.

# Forcing Relation for Product Games

$s\rho_{G \cup G'}^I X$  iff  $s\rho_G^I X$  or  $s\rho_{G'}^I X$ .

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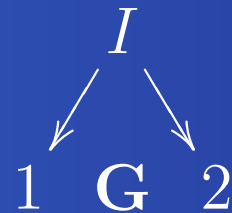
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$s\rho_{G^d}^{II} X$  iff  $s\rho_G^I X$ .

$s\rho_{G;G'}^i X$  iff  $\exists U : s\rho_G^i U$  and for each  $u \in \bigcup U$ ,  
 $u\rho_{G'}^i X$ .

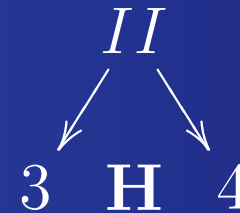
$s\rho_{G \times G'}^i X$  iff  $\exists T, \exists W : s\rho_G^i T$  and  $s\rho_{G'}^i W$  and  
 $X = \{t \cup w : t \in T \text{ and } w \in W\}$ .

# Forcing Relation for Product Games



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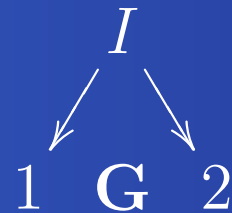
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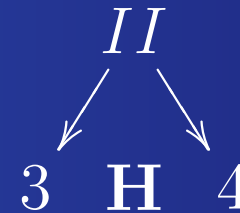
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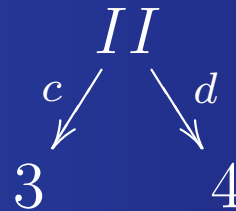
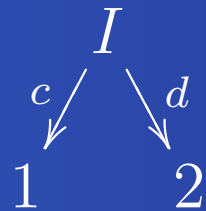
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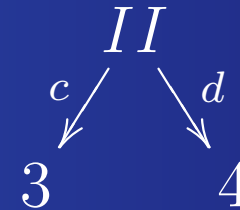
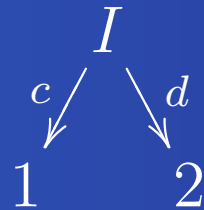
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# Prisoners' Dilemma (revisited)

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# Concurrent *DGL*

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Language:

$$\gamma := g \mid \phi? \mid \gamma; \gamma \mid \gamma \cup \gamma \mid \gamma^d \mid \gamma \times \gamma$$
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Conjunctive Game Model:  $\mathcal{M} = (S, \{\rho_g^i \mid g \in \Gamma\}, V)$

Semantics:

$\mathcal{M}, s \models \langle \gamma, i \rangle \phi$  iff there exists  $X : s \rho_\gamma^i X$  and

$\forall x \in \cup X : \mathcal{M}, x \models \varphi$

# Axioms

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+

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$$\langle \alpha \times \beta, i \rangle \phi \leftrightarrow \langle \alpha, i \rangle \phi \wedge \langle \beta, i \rangle \phi$$

# Completeness Theorem

# Completeness Theorem

- **Concurrent  $DGL$**  is sound and complete w.r.t. the class of all conjunctive game models. The logic is also decidable.

(van Benthem,   , Liu, 2007)

# *CDGL*-Bisimulation

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- *CDGL* formulas are invariant for *CDGL*-bisimulations.
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# Logic Games vs. Game Logics

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Open Question.....

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Possible Candidate -

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Open Question.....

**Possible Candidate** - *Branching quantifier Game Logic/ IF evaluation games of imperfect information*

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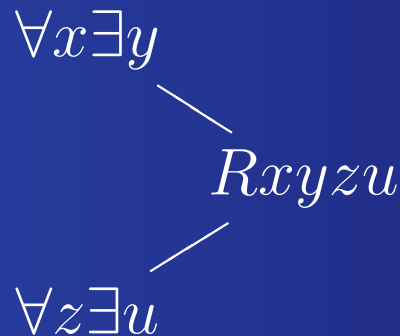
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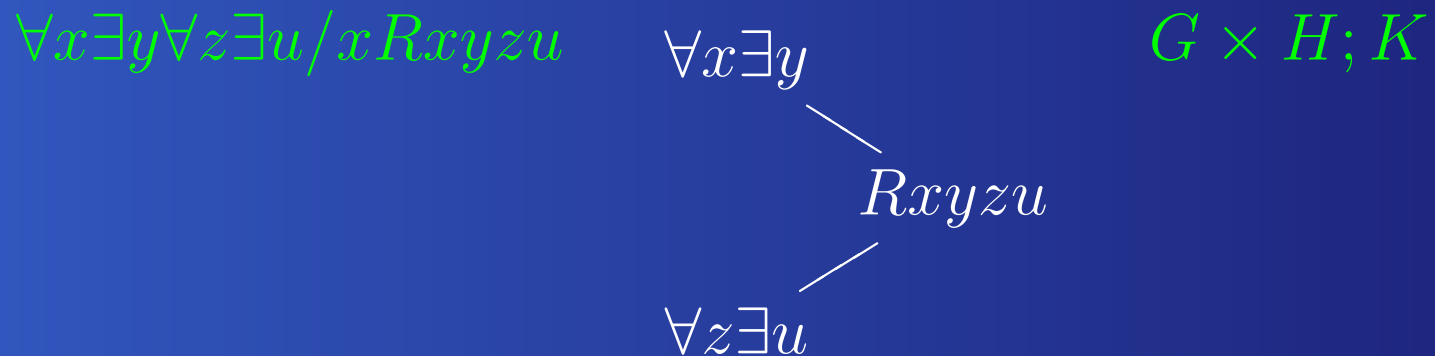
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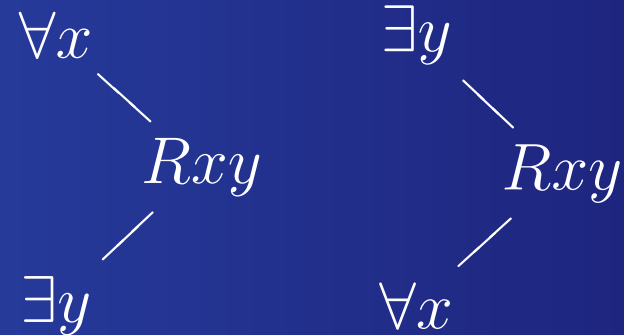


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$$(G, \vee, \wedge, -, \diamond, \times)$$

# Product Game Algebra

$$x \times x \approx x \quad (G12)$$

$$x \times y \approx y \times x \quad (G13)$$

$$(x \times y) \times z \approx x \times (y \times z) \quad (G14)$$

$$x \times (y \vee z) \approx (x \times y) \vee (x \times z) \quad x \times (y \wedge z) \approx (x \times y) \wedge (x \times z) \quad (G15)$$

$$\neg(x \times y) \approx \neg x \times \neg y \quad (G16)$$

$$(x \times y) \diamond (u \times v) = (x \diamond u) \times (y \diamond v) \quad (G17)$$

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Open Question.....

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explicit notions of strategies

# Some recent works on strategies

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# A research community

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Logic and Rational Interaction

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(<http://www.ilic.uva.nl/wordpress>)

Thanks for your time!